

THE GEOMETRY AND COMBINATORICS OF
 G -MODULES, AND WHERE TO FIND THEM

**Polynomial Expressions via
Borel-Weil-Bott and String
Parametrisation of Crystal Bases**

May 2, 2024

Contents

1	Introduction	3
2	Preparations in Algebraic Groups	6
2.1	First definitions	6
2.2	Lie algebras	8
2.3	Jordan-Chevalley decomposition	9
2.4	Structure theory of connected solvable groups	11
2.5	Definitions: reductive and semisimple	11
2.6	Tori of algebraic groups	11
2.7	Borel subgroups	13
2.8	Actions of the Weyl group	15
2.9	Structure theory of reductive groups	19
2.9.1	Regular and singular tori	19
2.9.2	Groups of semisimple rank 1	20
2.9.3	Reductive groups	20
3	Interlude: Indulgence in Infinitesimal Imprecision	25
3.1	G -modules and $\mathfrak{g}$ -modules	25
3.2	G -actions and vector fields on varieties	26
4	Polynomial Expressions for Irreducible Representations	29
4.1	Irreducible representations as line bundles L_λ : the Borel-Weil-Bott Theorem	30
4.2	$\Gamma(G/B, L_\lambda)$ as polynomials: the road map	31
4.3	Embedding into $\mathbb{C}(G/B)$ and what is preserved	32
4.4	Schubert varieties and Bott-Samelson varieties: first encounter	33
4.4.1	Bruhat decomposition	33
4.4.2	Schubert cells and Schubert varieties	34
4.4.3	Opposite Schubert cells	35
4.4.4	Bott-Samelson varieties	36
4.5	Global sections of L_λ are polynomials	37

5	Links to String Parametrisation of Crystal Bases	38
5.1	Valuations on polynomial rings	39
5.2	Crystal bases	39
	5.2.1 String parametrisation of crystal bases	40
5.3	Main theorem: highest exponents equal string parameters	43
5.4	More on Schubert and Bott-Samelson varieties	43
5.5	Computing string parameters along a filtration	46
5.6	Hurry now! The Proof!	48

Chapter 1

Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Its representation theory is well-understood in terms of its root systems (see any basic course on Lie algebras, for example [Hum72]): the irreducible finite-dimensional representations of $\mathfrak{g}$ are in 1-to-1 correspondence with its integral dominant weights. This can be done entirely algebraically, using constructions such as the universal enveloping algebra.

The **first question** we pose and investigate in this essay is the following: do we have a geometric perspective towards the irreducible finite-dimensional representations of $\mathfrak{g}$?

The natural place to start is a study of algebraic groups, as such groups are the most natural geometric objects associated with (or rather, giving rise to) Lie algebras. The representation theory of semisimple algebraic groups is in close parallel with that of semisimple Lie algebras:

Let G be a connected semisimple algebraic group over $\mathbb{C}$. Then there is again a 1-to-1 correspondence between finite-dimensional irreducible G -modules and dominant weights of G . Given a dominant weight λ , there is a unique finite-dimensional irreducible G -module V_λ of highest weight λ , and any such representation arises this way.

Given the somewhat subtle correspondence between algebraic groups and their Lie algebras, we will in fact approach the question above in the setup of algebraic groups. The precise correspondence between the representations of algebraic groups and Lie algebras is a little subtle. But we mention for now that a G -module is always a $\mathfrak{g}$ -module (see Section 3.1); and that in characteristic 0, the G -module is irreducible if and only if the induced $\mathfrak{g}$ -module also is (see [Hum75, Theorem 13.2]).

After some preparations in algebraic groups, we give an exposition of the amazing theorem of Borel-Weil-Bott in Section 4.1, providing a satisfactory answer to our first question. It states that for every dominant weight λ , there is a line bundle L_λ over a space G/B (called the flag variety), whose global sections are the irreducible G -module V_λ^* of lowest weight $-\lambda$. We give a construction of L_λ and an explicit description of the G -action on $\Gamma(G/B, L_\lambda)$, which is not

so easily found written down in the literature. We then makes sense of the statement that the induced actions by $\mathfrak{g} = \mathcal{L}(G)$ are *differentiation of sections*. This supplies a beautiful geometric perspective towards the otherwise abstract irreducible $\mathfrak{g}$ -modules.

In algebraic geometry, it is often the case that global sections of line bundles (i.e. locally free sheaves of rank 1) are naturally identified with polynomials. For example, on any scheme X , the trivial line bundle is isomorphic to the sheaf of regular functions $\mathcal{O}_X$ on X . So if $X = \operatorname{Spec} \mathbf{k}[x_1, \dots, x_n]$, then the space of global sections $\Gamma(X, \mathcal{O}_X)$ of $\mathcal{O}_X$ is simply the polynomial ring $\mathbf{k}[x_1, \dots, x_n]$. For a less trivial example, take $X = \mathbb{P}_{\mathbf{k}}^n = \operatorname{Proj} \mathbf{k}[x_0, \dots, x_n]$. The sheaf $\mathcal{O}_X(n) = \mathcal{O}_X(1)^{\otimes n}$ (where $\mathcal{O}_X(1)$ is the twisting sheaf) is a line bundle, and its global sections are naturally identified with the homogeneous polynomials of degree n in $\mathbf{k}[x_0, \dots, x_n]$.

This leads us to our **second question**: can we express global sections $\Gamma(G/B, L_\lambda)$ also as polynomials? We answer this question in Chapter 4. We approach this by choosing a suitable embedding of $\Gamma(G/B, L_\lambda)$ into $\mathbb{C}(G/B)$ and studying the Schubert varieties on G/B and their corresponding Bott-Samelson varieties. A more detailed overview of our arguments is given in Section 4.2.

Such polynomial expressions give us an abundance of concrete combinatorial data associated to the G -modules V_λ^* . Separately, the theory of crystal bases also provides combinatorial descriptions for the irreducible representations of G . We will give a precise definition of crystal bases in Section 5.2, but for now let us note that a crystal basis of a G -module is a basis consisting of weight vectors and satisfying some special combinatorial properties. The string parametrisation introduced by Littelmann in [Lit98] assigns to each basis element an integral coordinate in the Euclidean space such that the crystal basis is arranged into the integral points inside a polytope. Such a integral coordinate is called the string parameters of the element, and the polytope is called the string polytope.

Our **third subject of investigation** is the connection between these two combinatorial objects – the polynomial expressions and the string parameters arising in irreducible representations. In Chapter 5, we show that the highest exponent of the polynomial coincides with the string parameters of the vector it represents. Along the way, we see that $\mathfrak{g}$ -action on $\Gamma(G/B, L_\lambda)$ partially correspond to *differentiation of polynomials*.

One upshot of this connection is that string polytopes can be realised as Newton-Okounkov bodies for G/B , therefore inviting new perspectives from the latter into the fundamental theory of crystal bases. The theory of Newton-Okounkov bodies was inspired by, among his other ideas, A. Okounkov’s observation that log-concavity appears as a deep parallel between statistical physics and representation theory ([Oko00]). It has since been developed by authors such as R. Lazarsfeld, M. Musta a, K. Kaveh and A. G. Khovanskii. These authors associated convex bodies to various algebraic or geometric objects – for example, line bundles on varieties, semigroups, graded algebras – and exploited convex geometry for their study ([KK12; LM09]). The theory of Newton-Okounkov bodies is a vast story bringing together ideas across algebraic geometry, commutative algebra, representation theory and combinatorics. We, however, will

not have time to explore this in the present expository writing.

The essay is organised as follows:

In Chapter 2, we offer a rapid introduction to the theory of algebraic groups, highlighting concepts such as tori and Weyl groups and culminating in the root systems of reductive groups to emphasise parallels to the theory of Lie algebras. Chapter 3 discusses the transitions between G -modules and $\mathfrak{g}$ -modules, partially using techniques from differential geometry and Lie groups. In particular, we use this to see that when G acts by translating rational functions on a variety X , $\mathfrak{g}$ acts by differentiating functions. This is interesting on its own right and essential in what follows. In Chapter 4, we discuss the Borel-Weil-Bott theorem and polynomial expressions, settling the first two questions. Finally, Chapter 5 discusses the attendant combinatorics of polynomial expressions and compares it to string parametrisation of crystal bases.

The content of this essay is heavily inspired by the stimulating and well-written paper of Kiumars Kaveh [Kav15]. Although we have rearranged their presentation, Chapters 4 and 5 are essentially reproductions of this paper. Chapter 3 and the section on the Borel-Weil-Bott Theorem (Section 4.1) grew out of efforts to understand details omitted from [Kav15].

Chapter 2

Preparations in Algebraic Groups

As mentioned in the Introduction, one major reward of studying algebraic groups is to see that every irreducible representation of a semisimple algebraic group G over $\mathbb{C}$ can be realised as the sections of a line bundle over a certain variety G/B , with its Lie algebra acting by differentiations. This is the Borel-Weil-Bott theorem and is treated in the next chapter, after we establish the basic theory of algebraic groups.

In this chapter, we offer a condensed treatment of algebraic groups. The goal is not to provide complete details of the subject, but rather to rapidly develop the language needed for the appreciation of subsequent chapters. Using Weyl groups as a focal point, we bring together the notions of Borel subgroups and characters and cocharacters of tori. Towards the end, the structure theory of reductive groups gives a satisfactory picture of their root systems. Hopefully, this approach allow readers with previous exposure to Lie algebras or abstract root systems to quickly get a feel of algebraic groups.

The materials in this chapter are largely based on [Hum75] and [Bor91]. We work with the classical theory of algebraic varieties over an algebraically closed field. Although many technical geometric arguments are outsourced to references, familiarity with the standard theory of varieties is helpful. In this chapter **only**, we do *not* insist varieties are irreducible.

2.1 First definitions

We will briefly introduce the key concepts and state some initial properties of algebraic groups. Proofs are omitted; for details, see for example [Hum75, Section 7].

Definition 2.1.1. Let G be a variety (irreducible or not). If there are a point $e \in G$ and *morphisms of varieties* $\mu: G \times G \rightarrow G$ and $\iota: G \rightarrow G$ such that the

underlying set of G becomes a group (with e as the identity, μ as the group operation map, and ι as the inverse map), then we say G is an algebraic group.

For our purposes, we will always assume our algebraic groups are affine, unless stated otherwise.

A **closed subgroup** of G is a subgroup of G which is closed in the Zariski topology.

Algebraic groups abound, many of which are familiar from other contexts. In accordance with human instincts, we start with two simple but prototypical examples.

Example 2.1.2 (Additive group $\mathbf{G}_a$). The additive group $\mathbf{G}_a$ is the affine variety $\mathbf{A}^1 \cong \operatorname{Spec} k[t]$ together with $e = 0$, and μ and ι specified by the comorphisms $\mu^*: k[t] \rightarrow k[t] \otimes k[t] \cong k[t_1, t_2], t \mapsto t_1 + t_2$ and $\iota^*: k[t] \rightarrow k[t], t \mapsto -t$. One checks that on points these simply give the group law $\mu(x, y) = x + y$. $\diamond$

Example 2.1.3 (Multiplicative group $\mathbf{G}_m$). The additive group $\mathbf{G}_m$ is the affine variety $K^\times = \mathbf{A}^1 \setminus \{0\} \cong \operatorname{Spec} k[t, t^{-1}] \cong \operatorname{Spec} k[t]_t$ that is the distinguished open set of $\mathbf{A}^1 \cong \operatorname{Spec} k[t]$ where t does not vanish. The identity is $e = 1$, and μ is given by $\mu^*: k[t^\pm] \rightarrow k[t^\pm] \otimes k[t^\pm] \cong k[t_1^\pm, t_2^\pm], t \mapsto t_1 t_2$. $\diamond$

An important class of algebraic groups is matrix groups. The simplest is the general linear group.

Example 2.1.4 (General linear group $\operatorname{GL}(n, \mathbf{k})$). The set of $n \times n$ matrices can be viewed as the affine variety $\mathbf{A}^{n^2} \cong \operatorname{Spec} k[\{x_{ij}\}_{i,j=1,\dots,n}]$. Then the set of *invertible* $n \times n$ matrices is the distinguished open set corresponding to the non-vanishing of the determinant $\det(x_{ij})_{i,j}$, which is still an affine variety. This becomes a group under matrix multiplications and inverses: we can explicitly write down μ^* using the matrix multiplication formula

$$\mu^*: k[\{x_{ij}\}_{i,j}] \rightarrow k[\{x_{ij}\}_{i,j}] \otimes k[\{y_{ij}\}_{i,j}], x_{ij} \mapsto \sum_{k=1}^n x_{ik} \otimes y_{kj},$$

and similarly ι^* can be written down using Cramer's rule. $\diamond$

Other matrix groups are easily obtained by passing to closed subgroups. In fact, every algebraic group G is isomorphic to a closed subgroup of some $\operatorname{GL}(n, \mathbf{k})$ ([Hum75, Theorem 8.6]). This is called the linearisation of affine groups and is often used when proving results about algebraic groups.

There are obvious notions of morphisms and isomorphisms.

Definition 2.1.5. Let G, H be algebraic groups. A morphism of varieties $\varphi: G \rightarrow H$ is a **morphism of algebraic groups** if it is also a group homomorphism. ϕ is an isomorphism if it has two-sided inverses (that are morphisms of algebraic groups).

Proposition 2.1.6. *Kernels and images of morphisms are closed subgroups.*

There is a unique irreducible component of G which passes through the identity e . This is called the **identity component** of G and denoted by G° . We say that G is **connected** if $G = G^\circ$.

Proposition 2.1.7. *G° is normal and of finite index in G .*

Proposition 2.1.8. *Let $\varphi: G \rightarrow H$ be a morphism of algebraic groups. Then $\varphi(G^\circ) = \varphi(G)^\circ$.*

Given $x \in G$, we can define an action on $\mathbf{k}[G]$ called the right translation of functions, sending $f \in \mathbf{k}[G]$ to $\rho_x f$, with $(\rho_x f)(y) = f(yx)$. Then $\rho: G \rightarrow \mathrm{GL}(\mathbf{k}[G])$, $x \mapsto \rho_x$ is a group homomorphism. Similarly, define left translations of functions by $(\lambda_x f)(y) = f(x^{-1}y)$. These notions are useful in numerous contexts. To start with, we have

Lemma 2.1.9 (Membership criterion for closed subgroups). *Let H be a closed subgroup of G defined by the ideal $I \subset \mathbf{k}[G]$. Then $x \in H$ if and only if $\rho_x(I) \subset I$.*

A **representation** of G is a morphism of algebraic groups $\rho: G \rightarrow \mathrm{GL}(V)$. We also call V a G -module, with the map ρ suppressed from the notation.

Let H be a closed subgroup of G . The **quotient** G/H is a subtle construction in the theory of algebraic groups. We only mention the construction of the coset space G/H as a variety, but omit the group structure when H is normal. This construction relies on:

Theorem 2.1.10. *Let G, H be as above. Then there is a representation $\varphi: G \rightarrow \mathrm{GL}(V)$ and a one dimensional subspace L of V such that H is the stabiliser of L .*

Passing to the projectivisation $\mathbb{P}(V)$, $[L]$ becomes a point, and so G/H set-theoretically corresponds to the orbit of $[L]$, which is a variety. We take this to be the definition of the variety G/H .

2.2 Lie algebras

To each algebraic group is associated a object called its Lie algebra. Let G be an algebraic group over $\mathbf{k}$. Its Lie algebra, denoted $\mathcal{L}(G)$, is simply defined to be the tangent space $T_e G$ of G at the identity. Recall that if X is a variety and $x \in X$ a point on the variety, then $T_x X$ can be defined as $\mathrm{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbf{k})$, where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ (the stalk of $\mathcal{O}_X$ at x). An element of the tangent space should be viewed as a derivation of the germs at that point.

An equivalent definition of $\mathcal{L}(G)$ is as the space of **left-invariant derivations**

$$\mathcal{L}(G) = \{\delta \in \mathrm{Der} \mathbf{k}[G] \mid \delta \lambda_x = \lambda_x \delta \quad \forall x \in G\}.$$

In particular, this definition equips $\mathcal{L}(G)$ with a Lie bracket: $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$.

It turns out that $\mathcal{L}(G)$ retains much of the information of G . Its introduction makes available infinitesimal techniques in the study of G and is crucial in the development of the theory.

Lie algebras have good functorial properties.

Given a morphism of algebraic groups $\varphi: G \rightarrow H$, taking its differential at the identity gives a Lie algebra homomorphism $d\varphi_e: \mathfrak{g} \rightarrow \mathfrak{h}$.

When H is a closed subgroup of G , the differential of the inclusion is in fact injective. Hence $\mathfrak{h}$ can be viewed as a Lie subalgebra of $\mathfrak{g}$.

The adjoint map of G is crucial going forward. Let $x \in G$. Then conjugation by x gives a group automorphism $\text{Int } x: G \rightarrow G, y \mapsto xyx^{-1}$. Its differential is denoted $\text{Ad } x: \mathfrak{g} \rightarrow \mathfrak{g}$. This is in fact an automorphism of $\mathfrak{g}$, by noting $(\text{Int } x) \circ (\text{Int } x^{-1}) = \text{id}_G$, whence $\text{Ad}(x) \circ \text{Ad}(x^{-1}) = \text{id}_{\mathfrak{g}}$. This makes $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ a group homomorphism, called the adjoint representation of G . In fact, it is even a morphism of *algebraic* groups, though we will not prove this.

2.3 Jordan-Chevalley decomposition

The analysis of semisimple and unipotent elements forms a cornerstone of the theory of algebraic groups. The starting point of this is the Jordan-Chevalley decomposition, whose additive version is well-known in linear algebra:

Theorem 2.3.1 (Additive Jordan-Chevalley decomposition). *Let V be a finite-dimensional vector space over an algebraically closed field $\mathbf{k}$, and $x \in \text{End}(V)$. Then there are unique elements $x_s, x_n \in \text{End}(V)$ such that $x = x_s + x_n$, x_s is semisimple (i.e. diagonalisable), x_n is nilpotent (i.e. $x_n^m = 0$ for some m), and $x_s x_n = x_n x_s$. Moreover, if x stabilises a subspace $W \subset V$, then so do x_s, x_n .*

Remark. Existence is immediate from the Jordan normal form of x . ∇

If now $x \in \text{GL}(V)$ is invertible, 0 is not an eigenvalue, so that x_s is also invertible. We can rewrite the additive decomposition of x as a multiplicative decomposition:

$$x = x_s + x_n = x_s(1 + x_s^{-1}x_n) = x_s x_u,$$

where we define $x_u = 1 + x_s^{-1}x_n$. Then x_u is also invertible. In fact, x_u and x_s commute. This process is reversible, so such decomposition is unique.

So we have:

Theorem 2.3.2 (Multiplicative Jordan-Chevalley decomposition). *Let V be a finite-dimensional vector space over an algebraically closed field k , and $x \in \text{GL}(V)$. Then there are unique elements $x_s, x_u \in \text{GL}(V)$ such that $x = x_s x_u = x_u x_s$, x_s is semisimple, x_u is unipotent. Moreover, if x stabilises a subspace $W \subset V$, then so do x_s, x_u .*

We need a notion of Jordan-Chevalley decomposition even when V is infinite-dimensional. Let $x \in \text{GL}(V)$. Suppose that V is the union of finite-dimensional subspaces stable under x . Then we can perform the Jordan-Chevalley decomposition on each such subspace, which agrees on overlapping subspaces by uniqueness. Hence we can glue together the semisimple (resp. unipotent) parts of x on

each such subspace to obtain a member of $\mathrm{GL}(V)$ which we call the semisimple (resp. unipotent) part of x . In particular, we will apply this to $\rho_x \in \mathrm{GL}(\mathbf{k}[G])$ for $x \in G$, where G is an arbitrary algebraic group.

We would like to know how to do Jordan decompositions in any algebraic group G . An obvious place to start is to embed G as a closed subgroup of some $\mathrm{GL}(n, \mathbf{k})$, then do the Jordan decomposition inside $\mathrm{GL}(n, \mathbf{k})$. It is not clear whether this is independent of the choice of the embedding, but there is a bigger problem: do the semisimple and unipotent parts of $x \in G$ inside $\mathrm{GL}(n, \mathbf{k})$ even lie in G ? The answer is no in general for subgroups of $\mathrm{GL}(n, \mathbf{k})$, but fortunately this is true when the subgroup is *closed*.

Proposition 2.3.3. *Let G be a closed subgroup of $\mathrm{GL}(n, \mathbf{k})$, and let $x \in G$. Then $x_s, x_u \in G$, where $x = x_s x_u$ is the Jordan decomposition in $\mathrm{GL}(n, \mathbf{k})$.*

Proof. Suppose G is defined by the ideal I . Using Lemma 2.1.9, it suffices to know that ρ_{x_s} and ρ_{x_u} preserve I . We know that ρ_x preserves I since $x \in G$. Hence, by the last part of Theorem 2.3.2, it is enough to know that ρ_{x_s} and ρ_{x_u} are the semisimple and unipotent parts of ρ_x . This is the next lemma. $\square$

Lemma 2.3.4. *Let $G = \mathrm{GL}(n, \mathbf{k})$ and $x \in G$. Then ρ_{x_s} and ρ_{x_u} are the semisimple and unipotent parts of ρ_x .*

Proof. [Hum75, Proposition 15.2]. $\square$

The statement of this lemma also inspires an intrinsic definition of semisimple and unipotent elements in arbitrary algebraic groups.

Definition 2.3.5. Let G be an algebraic group. An element $x \in G$ is **semisimple** (resp. **unipotent**) if ρ_x is semisimple (resp. unipotent).

We obtain the Jordan-Chevalley decomposition in arbitrary algebraic groups.

Theorem 2.3.6. *Let G be an algebraic group and let $x \in G$. Then there are unique commuting elements s and u such that $x = su$, s is semisimple, and u is unipotent.*

Proof. Embed G as a closed subgroup of some $\mathrm{GL}(n, \mathbf{k})$. Let $x \in G$ and decompose $x = su$ in $\mathrm{GL}(n, \mathbf{k})$.

Then Proposition 2.3.3 shows that $s, u \in G$, and Lemma 2.3.4 shows that ρ_s and ρ_u are semisimple and unipotent, respectively. Hence s is semisimple, u is unipotent by definition.

Now $\rho_s \rho_u$ is the unique decomposition of ρ_x in $\mathrm{GL}(\mathbf{k}[G])$, so by the faithfulness of ρ , $x = su$ is the unique decomposition in G . $\square$

It is now possible to intrinsically define the subsets $G_s = \{x \in G \mid x = x_s\}$ and $G_u = \{x \in G \mid x = x_u\}$ of semisimple and unipotent elements in an arbitrary algebraic group G .

The language described here is essential in what follows.

2.4 Structure theory of connected solvable groups

Solvability of algebraic groups is defined in the same way as abstract groups. Recall that a group G is solvable if the derived series inductively defined by $G^{(0)} = G, G^{(n)} = (G^{(n-1)}, G^{(n-1)})$ ($n \geq 1$) terminates in the identity.

A more stringent condition is nilpotency. The **descending central series** of G is defined inductively by $\mathcal{D}^0 G = G, \mathcal{D}^n G = (G, \mathcal{D}^{n-1} G)$. Put $G^\infty = \cap \mathcal{D}^i G$.

Theorem 2.4.1. *Let G be a connected solvable algebraic group. All maximal tori of G are conjugate under G^∞ . If T is a maximal torus (see the next section), then $G = T \ltimes G_u$.*

Proof. [Hum75, Theorem 19.3] □

It can be checked that the upper triangular group $T(n, \mathbf{k})$ is connected and solvable (see [Hum75, Section 17.3]). The intuition for the above theorem is that $T(n, \mathbf{k})$ decomposes as the diagonal matrices and the unipotent matrices (with 1's on the diagonal). This will make more sense in subsequent sections. We will use the above result many times.

2.5 Definitions: reductive and semisimple

Let G be an algebraic group. Its **radical** $R(G)$ is the unique largest *connected normal* solvable subgroup of G . This is well-defined (see [Hum75, Lemma 17.3]). Its unipotent radical $R_u(G)$ is the unipotent part of $R(G)$.

We say that G is **semisimple** if $R(G)$ is trivial, and that G is **reductive** if $R_u(G)$ is trivial. Note that being semisimple is stronger and implies being reductive.

2.6 Tori of algebraic groups

Just as in the theory of Lie algebras, the “toral” (“commutative and semisimple”) parts of algebraic groups are a fruitful object of study, leading to later notions such as the Weyl group.

Definition 2.6.1. An algebraic group is a torus if it is isomorphic to $\underbrace{\mathbf{G}_m \times \cdots \times \mathbf{G}_m}_{n \text{ times}}$

(which is isomorphic to the diagonal group $D(n, \mathbf{k})$).

The notion of characters is especially important for studying tori as well as the larger groups in which they sit. A character ξ of an algebraic group G is a morphism of algebraic groups $\xi: G \rightarrow \mathbf{G}_m$. We can multiply or divide characters by performing the operation pointwise in $\mathbf{G}_m$. In this way the set $X(G)$ of characters of G forms an abelian group. It is not hard to check that $X(D(n, \mathbf{k})) \cong \mathbb{Z}^n$ by considering the co-morphisms of characters $\xi^*: k[t^\pm] \rightarrow k[x_1^\pm, \dots, x_n^\pm]$.

We would like a more intrinsic characterisation of tori. For this purpose and for obtaining some important results regarding tori, we must turn to the more general context of diagonalisable groups. We will highlight the important results but not dwell on the details too much.

Definition 2.6.2. An algebraic group is diagonalisable if it is isomorphic to a closed subgroup of $D(n, \mathbf{k})$.

We have an alternative, more intrinsic characterisation:

Lemma 2.6.3. *G is diagonalisable if and only if it is commutative and consists of semisimple elements.*

Proof. One direction is obvious: if G is diagonalisable, it is certainly commutative. The isomorphism to a closed subgroup into $D(n, \mathbf{k})$ induces a closed embedding into the $GL(n, \mathbf{k})$ in which $D(n, \mathbf{k})$ sits. Recall that any closed embedding of G into a general linear group can be used to give the (same) Jordan-Chevalley decomposition in G . Here since the image of G sits inside $D(n, \mathbf{k})$, G consists of semisimple elements.

Conversely, suppose G is commutative and consists of semisimple elements. We first embed G as a closed subgroup of some $GL(n, \mathbf{k})$. Then we invoke the fact from linear algebra that any set of commuting diagonalisable maps can be *simultaneously* diagonalised. So G is conjugate, hence isomorphic to a subgroup of $D(n, \mathbf{k})$. $\square$

Corollary 2.6.4. *Closed subgroups and homomorphic images of a diagonalisable group are diagonalisable.*

Proof. Such groups are commutative and consist of semisimple elements. $\square$

Theorem 2.6.5. *A connected diagonalisable group is a torus.*

Proof. [Hum75, Theorem 16.2]. $\square$

Corollary 2.6.6. *Homomorphic images of tori are tori.*

Proof. The image of a torus is connected because the torus is. It is diagonalisable by Corollary 2.6.4. Now conclude by the theorem. $\square$

Now let T be a torus in G . Recall the morphism of algebraic groups $\text{Ad}: G \rightarrow GL(\mathfrak{g})$. By Corollary 2.6.6, $\text{Ad } T$ is a torus in $GL(\mathfrak{g})$ and acts on $\mathfrak{g}$ via commuting semisimple elements. Then again by simultaneous diagonalisability, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\Phi \subset X(T)$,

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g} \mid (\text{Ad } t)(v) = \alpha(t).v \text{ for all } t \in T\}$$

$$\mathfrak{g}^T = \{v \in \mathfrak{g} \mid (\text{Ad } t)(v) = v \text{ for all } t \in T\}.$$

We also write $\Phi(T, G)$ for Φ .

Definition 2.6.7. A torus $T \subset G$ of maximal dimension is called a maximal torus of G .

When T is maximal, each element of $\Phi(T, G)$ is called a **root**.

We are most interested in the case when T is a *maximal* torus, and when G is *reductive*. In this case, we have

Proposition 2.6.8. *If G is reductive and T is a maximal torus, then $\mathfrak{g}^T = \mathfrak{t}$, and each $\mathfrak{g}_\alpha$ is one-dimensional.*

Proof. This is later shown in Theorems 2.9.9 and 2.9.12. $\square$

Moreover, in this case we in fact know that Φ is a root system ([Hum75, Section 26.2]), giving a satisfactory characterisation of the Lie algebras of reductive groups. This picture becomes sharper as we introduce Weyl groups from the point of view of algebraic groups. But first, let us discuss Borel subgroups, indispensable for both this purpose and for further development of the theory.

2.7 Borel subgroups

Let G be an algebraic group. We can leverage our understanding of connected solvable groups to analyse general algebraic groups. Therefore, we define the Borel subgroups:

Definition 2.7.1. A maximal closed connected solvable subgroup B of G is called a Borel subgroup.

Later on, we will discover that Borel subgroups are also the ground on which to realise the combinatorial aspects of root systems in algebraic groups.

Note that because of the *connectedness* requirement in the definition, the Borel subgroups of G and G° will coincide. *From now on, we will assume G is connected.*

Example 2.7.2. The subgroup $B = T(n, \mathbf{k})$ of upper triangular matrices in $\mathrm{GL}(n, \mathbf{k})$ is a Borel subgroup. It is clearly closed and connected. It is easy to see that (B, B) is contained in $U(n, \mathbf{k})$, and with a bit of effort one can see $U(n, \mathbf{k})$ is solvable (e.g., see [Hum75, Section 17.3]). So T is solvable. It remains to see that it is maximal. Lie-Kolchin Theorem states that any connected solvable subgroup of $\mathrm{GL}(n, \mathbf{k})$ is conjugate to a subgroup of T . This shows that B is maximal. $\diamond$

Example 2.7.3. Similarly, one can see that $T(n, \mathbf{k}) \cap \mathrm{SL}(n, \mathbf{k})$ is a Borel subgroup of $\mathrm{SL}(n, \mathbf{k})$. $\diamond$

The fundamental geometric property of Borel subgroups is the following.

Theorem 2.7.4. *Let B be a Borel subgroup of G . Then G/B is a projective variety. In particular, it is complete (i.e. universally closed; this is the analog of compactness in algebraic geometry).*

Remark. In fact, G/B is the “largest quotient of G ” that is a projective variety. ∇

Hence the following geometric fact is relevant, which we will apply time and again.

Theorem 2.7.5 (Fixed Point Theorem). *Let G be a connected solvable algebraic group, and suppose it acts on a nonempty complete variety. Then the G -action has a fixed point in X .*

Denote by $\mathfrak{B}$ the set of all Borel subgroups of G . The fixed point theorem 2.7.5 gives a strong characterisation of $\mathfrak{B}$.

Theorem 2.7.6. *All Borel subgroups of G are conjugate.*

Proof. Let B and B' be two Borel subgroups of G .

Then B' acts by left multiplication on the complete variety G/B . By the fixed point theorem 2.7.5, there is a point xB such that $B'xB = xB$.

In other words, $x^{-1}B'x \subset B$. Since conjugation by x is an isomorphism of algebraic groups, $x^{-1}B'x$ is also a Borel subgroup. But then by maximality, we have equality $x^{-1}B'x = B$. $\square$

Corollary 2.7.7. *All maximal tori T of G are conjugate.*

Proof. Let T, T' be two maximal tori. They are connected and solvable, hence contained in some Borel subgroups B, B' , respectively. Then some conjugation carries B' to B , and so carries T' to a maximal torus in B . Now the result follows since all maximal tori of B are conjugate (Theorem 2.4.1). $\square$

The variety G/B is so important that we give it a name: G/B is called the **flag variety** of G . Thanks to the conjugacy result just proved, it makes no essential difference which B is chosen.

Remark. G/B is called the flag variety because of the case $G = \mathrm{GL}(n, \mathbf{k})$. In this case, right actions by upper-triangular matrices give just enough column actions to put each matrix into the column echelon form. Then taking the span of first k columns ($0 \leq k \leq n$) gives a filtration of vector spaces, which is called a flag (cf. [WY23]):

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbf{k}^n.$$

∇

The action of Borel subgroups on G/B is important. In this case, the fixed point theorem can be sharpened: the fixed point is in fact unique. This is a corollary of a more subtle result about the normalisers of Borel subgroups:

Theorem 2.7.8. *Let B be a Borel subgroup of G . Then $B = N_G(B)$.*

Then we can show:

Corollary 2.7.9. *Fix a Borel subgroup B and let G/B be the flag variety. If S is a Borel subgroup, the action of S on G/B by left multiplication has a unique fixed point.*

Proof. We already know that it has a fixed point xB , i.e. $xBx^{-1} = S$ by the proof of Theorem 2.7.6.

Let yB also be a fixed point. Then $yBy^{-1} = S$.

Hence $x^{-1}yB(x^{-1}y)^{-1} = B$, and $x^{-1}y \in N_G(B) = B$, so $xB = yB$. $\square$

Thus we can identify the set $\mathfrak{B}$ with the flag variety G/B (for a fixed B), by sending a Borel $S \in \mathfrak{B}$ to its unique fixed point $xB \in G/B$. The equivalence that xB is a fixed point if and only if $xBx^{-1} = S$ makes it clear that this map is both injective and surjective.

The **natural action of G on $\mathfrak{B}$** via conjugation ($S \mapsto xSx^{-1}$) will be important later on. For now, observe that under the above identification, the conjugation action on $\mathfrak{B}$ goes over to the natural action of G on G/B by left multiplication ($yB \mapsto xyB$): indeed, if S fixes yB , then xSx^{-1} fixes xyB .

We sometimes also want to consider not all of $\mathfrak{B}$, but the set $\mathfrak{B}^H$ of **Borel subgroups containing a given subgroup H** . The suggestive notation is appropriate because $\mathfrak{B}^H$ is precisely those Borel subgroups that are fixed under conjugation by H : if H is contained in B , then certainly H stabilises (normalises) B ; conversely, if H normalises B , then $H \subset N_G(B) = B$.

Borel subgroups also give us a handle on the radical $R(G)$ of G .

Proposition 2.7.10.

$$R(G) = \left(\bigcap_{B \in \mathfrak{B}} B \right)^\circ$$

Proof. Since each Borel B is solvable, so is their intersection. This intersection is normal because all Borel subgroups are conjugates. Passing to the identity component gives connectedness, while preserving solvability and normality.

So the RHS is connected solvable normal. It is maximal amongst all such groups because any such group S is contained in one Borel subgroup, and hence all Borel subgroups by the conjugacy of Borel subgroups and the normality of S . $\square$

2.8 Actions of the Weyl group

In the study of semisimple Lie algebras, the Weyl group is a fundamental combinatorial object associated to the root system of the Lie algebra.

Now let G be an algebraic group and fix a maximal torus T . In Section 2.6, we defined $\Phi = \Phi(T, G)$ and claimed that Φ is root system when G is reductive (recall that this was done by considering the torus actions in $\mathcal{L}(G)$). Hence G should also have a Weyl group.

In fact, we can intrinsically define the Weyl group of any algebraic group without appealing to Φ being a root system (and it may not be if G is not reductive).

In this section, we will give this definition and, by considering its actions on various objects, realise that it behaves in familiar ways.

Definition 2.8.1. Let T be a maximal torus of an algebraic group G . The **Weyl group** of G relative to T is defined to be $W = W(G, T) := N_G(T)/C_G(T)$.

Since all maximal tori are conjugate (Corollary 2.7.7), it is immaterial which maximal torus is chosen; they all give isomorphic Weyl groups.

Proposition 2.8.2. W is a finite group.

Proof. It follows from the “rigidity of tori” ([Hum75, Corollary 16.3]) that $N_G(T)^\circ = C_G(T)^\circ$. Then, since the identity component of any algebraic group has finite index in the group (Proposition 2.1.7), applying this to $N_G(T)$ and $C_G(T)$ respectively implies $|W| = [N_G(T) : C_G(T)]$ is also finite. $\square$

We have a first action of W on the set $\mathfrak{B}^T$ of Borel subgroups containing a fixed maximal torus T . As $N := N_G(T)$ normalises T , it preserves $\mathfrak{B}^T$. On the other hand,

Lemma 2.8.3. $C := C_G(T)$ is contained in every $B \in \mathfrak{B}^T$. In particular, it acts trivially on $\mathfrak{B}^T$.

Proof. Since C is connected and nilpotent ([Hum75, Sections 22.3 and 21.4]), C is contained in some Borel subgroup $B \in \mathfrak{B}^T$. If $B' = xBx^{-1}$ is another Borel subgroup in $\mathfrak{B}^T$ (B' can be written this way since all Borel subgroups are conjugates 2.7.6), then $xCx^{-1} \subset B'$. Now T and xTx^{-1} are both maximal tori in B' , so conjugate by some $y \in B'$ (Theorem 2.4.1). Hence $C = C_G(yxTx^{-1}y^{-1}) = y(xCx^{-1})y^{-1} \subset B'$. $\square$

Therefore, $W = N/C$ acts on $\mathfrak{B}^T$. This offers a first justification for defining W this way.

Theorem 2.8.4. Let $T, W, \mathfrak{B}^T$ be as above. Then W permutes $\mathfrak{B}^T$ simply transitively. In particular, $|\mathfrak{B}^T| = |W|$ is finite.

Proof. [Hum75, Section 24.1]. $\square$

We can further sharpen this view. But now let us first introduce **the notion dual to characters**.

Let T be a torus. A **one parameter subgroup** (or **1-psg** for short) of T is a morphism of algebraic groups $\lambda: \mathbf{G}_m \rightarrow T$. We also call λ a **cocharacter**. The set of all cocharacters becomes an abelian group in the obvious way, denoted $Y(T)$.

If $\xi \in X(T), \lambda \in Y(T)$, the composition $\xi \circ \lambda$ is an element of $X(\mathbf{G}_m) \cong \mathbb{Z}$. So this defines a natural pairing $\langle \cdot, \cdot \rangle: X(T) \times Y(T) \rightarrow \mathbb{Z}$, sending $\langle \xi, \lambda \rangle$ to the integer identified with $\xi \circ \lambda$. It can be checked that this is a perfect pairing.

We now describe the actions of W on $X(T)$ and $Y(T)$.

Let $n \in N_G(T)$, $\alpha \in X(T)$. Define $n \cdot \alpha$ by $(n \cdot \alpha)(t) = \alpha(n^{-1}tn)$. This action is trivial if $n \in C_G(T)$. Hence this descends to a W -action on $X(T)$.

In turn, W permutes $\Phi(T, G)$ via $(\text{Ad } n)\mathfrak{g}_\alpha = \mathfrak{g}_{w \cdot \alpha}$, where n represents $w \in W$. To see this, consider $(\text{Ad}(n^{-1}tn))\mathfrak{g}_\alpha = \alpha(n^{-1}tn)\mathfrak{g}_\alpha = (w \cdot \alpha)(t)\mathfrak{g}_\alpha$, so $(\text{Ad}(t))(\text{Ad } n)\mathfrak{g}_\alpha = (w \cdot \alpha)(t)(\text{Ad } n)\mathfrak{g}_\alpha$.

Similarly, W acts on $Y(T)$ by setting $(w \cdot \lambda)(a) = n\lambda(a)n^{-1}$, where n represents $w \in W$.

One checks that the perfect pairing is W -equivariant: $\langle \alpha, \lambda \rangle = \langle w \cdot \alpha, w \cdot \lambda \rangle$.

Definition 2.8.5. A 1-psg $\lambda \in Y(T)$ is called regular if $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in \Phi$. The collection of regular 1-psgs is denoted $Y(T)_{\text{reg}}$.

Remark. Later we will define the notion of regular tori in 2.9.1 and will show that a 1-psg $\lambda \in Y(T)$ is regular if and only if the torus $\lambda(\mathbf{G}_m)$ is regular. ∇

Since $\langle \alpha, \lambda \rangle = \langle w \cdot \alpha, w \cdot \lambda \rangle$, we see that W permutes $Y(T)_{\text{reg}}$.

We now describe a construction to assign a Borel subgroup $B(\lambda) \in \mathfrak{B}^T$ to each regular 1-psg λ . This assignment makes compatible the Weyl group actions on $Y(T)_{\text{reg}}$ and on $\mathfrak{B}^T$. Later, we also see this leads to the familiar notion of Weyl chambers.

Recall that G/B is a complete (projective!) variety which can be identified with the set $\mathfrak{B}$ of all Borel subgroups, and the G -actions on G/B and $\mathfrak{B}$ are compatible.

The idea of the construction is as follows. Fix a $\lambda: \mathbf{G}_m \rightarrow T$ and a point $x \in G/B$. Using the action on G/B , construct an associated map $f_{\lambda, x}: \mathbf{G}_m \rightarrow G/B$ with $f_{\lambda, x}(t) = \lambda(t) \cdot x$. In fact, $f_{\lambda, x}$ is a morphism of varieties (see below). Since G/B is complete (think compact for analogy), by the general theory of varieties we have that $f_{\lambda, x}$ extends to $\mathbb{P}^1 \rightarrow G/B$. In particular, we get a point $f_{\lambda, x}(\infty) \in G/B$. Up to starting with a suitable x , the point $f_{\lambda, x}(\infty)$ is the Borel subgroup we would like to call $B(\lambda)$. Actually, the initial choice of x doesn't matter, as we make precise in the **uniqueness** part of the full construction:

Proposition 2.8.6. *Fix a regular 1-psg $\lambda: \mathbf{G}_m \rightarrow T$. There exists a unique point $x(\lambda) \in G/B$ such that $f_{\lambda, x}(\infty) = x(\lambda)$ for all x in some open neighborhood (dense since G/B irreducible) of $x(\lambda)$. The corresponding Borel subgroup is denoted $B(\lambda)$, and it contains T .*

Proof. We first settle the **uniqueness**: suppose there are points $x(\lambda), x'(\lambda)$ with open neighborhoods U, U' satisfying the above statement. Since $U \cap U'$ is non-empty by the irreducibility of X , picking $x \in U \cap U'$ would force $x(\lambda) = f_{\lambda, x}(\infty) = x'(\lambda)$.

Recall that (Theorem 2.1.10) we realise $X = G/B$ as a projective variety by choosing a representation $\rho: G \rightarrow \text{GL}(V)$ and identifying G/B with the orbit of a point $\mathbb{P}(V)$. Call $\pi: V \setminus \{0\} \rightarrow \mathbb{P}(V)$ the canonical projection. By replacing V with $\pi^{-1}(X)$ if necessary, we can assume X lies in no hyperplane of $\mathbb{P}(V)$.

Now ρ sends $T \subset G$ into a torus in $\text{GL}(V)$, so V has a basis consisting of eigenvectors $v_1, \dots, v_n$ corresponding to some characters $\gamma_1, \dots, \gamma_n \in X(T)$. If

we set $m_i = \langle \gamma_i, \lambda \rangle$, then λ acts via

$$\lambda(t) \cdot \left(\sum a_i v_i \right) = \sum a_i t^{m_i} v_i.$$

Assume we have ordered the basis from the start such that $m_1 \geq \dots \geq m_n$. Assume also $m_1 = \dots = m_r > m_{r+1}$.

Now fix a $v_0 = \sum a_i v_i$ (which we will later show doesn't matter) with $[v_0] \in X$, $a_1 \neq 0$. This is possible because we have arranged that X does not lie in the hyperplane $\mathbb{P}(W)$, where W is spanned by $v_2, \dots, v_n$.

Then we can see that as $t \rightarrow \infty$, $f_{\lambda, [v_0]}(t) = \lambda(t) \cdot [v_0]$ is dominated by the exponents $m_1, \dots, m_r$, and $\lambda(\infty) \cdot [v] = [a_1 v_1 + \dots + a_r v_r]$. In fact, since λ is regular, one can show that $r = 1$, so that $\lambda(\infty) \cdot [v_0] = [v_1]$. For rigorous justifications of this paragraph, see [Bor91, Sections 13.3, 13.8] or [Hum75, Sections 25.1, 25.4].

Define $x(\lambda) = [v_1]$. The above calculation together with $r = 1$ shows that $\lambda(\infty) \cdot [v] = [v_1] = x(\lambda)$ whenever $[v] \notin \mathbb{P}(W)$. So $X \setminus (X \cap \mathbb{P}(W))$ is a (non-empty) open neighborhood of $x(\lambda)$ as required. Moreover, the Borel subgroup corresponding to $x(\lambda)$, denoted $B(\lambda)$, lies in $\mathfrak{B}^T$ (i.e. contains T), since T clearly fixes $x(\lambda) = [v_1]$. $\square$

Next we see that the assignment $\lambda \rightarrow B(\lambda)$ is W -equivariant. As above, let $w \in W$ and $n \in N_G(T)$ a representative. We need to find out which point $x(w \cdot \lambda)$ is produced by the procedure above. Now, n permutes the eigenspaces of T in V , and recalling the W -actions on $X(T)$, it is easy to verify that $n \cdot v_i$ is an eigenvector with weight $n \cdot \gamma_i$. Now since m_1 was the largest amongst m_i 's and $m_1 = \langle \gamma_1, \lambda \rangle = \langle n \cdot \gamma_1, n \cdot \lambda \rangle$, we see that $n \cdot v$ is the dominating vector in the proof above, and $x(w \cdot \lambda) = [n \cdot v]$. But $[n \cdot v]$ corresponds to the Borel subgroup $nB(\lambda)n^{-1}$, so $B(w \cdot \lambda) = w \cdot B(\lambda)$, as desired.

Since W acts transitively on $\mathfrak{B}^T$, the W -equivariance shows in particular that $\lambda \rightarrow B(\lambda)$ is surjective. One major gain of this construction is the definition of **Weyl chambers**. Given $B \in \mathfrak{B}^T$, define the Weyl chamber $\mathfrak{C}(B) = \{\lambda \in Y(T)_{\text{reg}} \mid B = B(\lambda)\}$. Then the Weyl chambers are in bijection with $\mathfrak{B}^T$, and W permutes Weyl chambers simply transitively.

Summary

This should be reminiscent of Weyl chambers in abstract root systems (or from Lie algebras). Recall that in the context of root systems, a (regular) cocharacter λ specifies a root basis (by taking the indecomposable roots on the positive side of the hyperplane defined by λ). Cocharacters in the same Weyl chamber give rise to the same root basis. Hence we should think of a Borel subgroup as a choice of root basis, and in particular it would specify a set of positive roots. The W -actions on $X(T), Y(T)$ revolve around this construction. We will see this more clearly in the next section.

2.9 Structure theory of reductive groups

In this section, our development is somewhat closer to [Bor91].

Let G be an algebraic group.

2.9.1 Regular and singular tori

Definition 2.9.1. A torus $S \subset G$ is called **regular** if $\mathfrak{B}^S$ is finite. Otherwise, it is called **singular**.

Note that since W is a finite group acting simply transitively on $\mathfrak{B}^T$ for any maximal torus T , maximal tori are regular.

Let $I(T)$ be the identity component of $\bigcap_{B \in \mathfrak{B}^T} B$. Then $I(T)$ is a closed subgroup of G normalised by T , since each B is closed and conjugation by T permutes $\mathfrak{B}^T$.

Consider this in the more general context where H is a closed subgroup normalised by T . Then $\mathfrak{h} := \mathcal{L}(H)$ is also stable under $\text{Ad } T$. So for each $\alpha \in \Phi(G, T)$, we can split $\mathfrak{g}_\alpha = \mathfrak{h}_\alpha \oplus \mathfrak{g}'_\alpha$ for some complement $\mathfrak{g}'_\alpha \subset \mathfrak{g}_\alpha$ of $\mathfrak{h}_\alpha = \mathfrak{h} \cap \mathfrak{g}_\alpha$. Define $\Phi(T, G/H)$ to be the roots $\alpha \in \Phi(T, G)$ such that $\mathfrak{g}'_\alpha$ – the roots outside of H .

Then

$$\mathfrak{g} = (\mathfrak{g}^T + \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi(T, G/H)} \mathfrak{g}'_\alpha.$$

Take $H = I(T)$. Then $G^T \subset I(T)$ by Lemma 2.8.3, so $\mathfrak{g}^T \subset \mathcal{L}(I(T)) = \mathfrak{h}$. Define $\Psi(T, G) = \Phi(T, G/I(T))$. Then the above simplifies to

$$\mathfrak{g} = \mathcal{L}(I(T)) \oplus \bigoplus_{\alpha \in \Psi(T, G)} \mathfrak{g}'_\alpha.$$

For now, one reason for introducing $I(T)$ is that Ψ gives an effective description of all singular tori occurring in G . Let $\alpha \in \Psi$, and let $T_\alpha = (\ker \alpha)^\circ \subset T$. Then T_α is a connected subgroup of T , hence a torus (of codimension 1). Then we have:

Proposition 2.9.2. *As before fix a maximal torus T and let $S \subset T$ be a subtorus. Then S is singular if and only if $S \subset T_\alpha$ for some $\alpha \in \Psi$.*

Proof. This is not hard, but see [Hum75, Proposition 24.3]. We require the following lemma. $\square$

Lemma 2.9.3. [Hum75, Proposition 24.2] *Let S be a torus in G . Then S is regular if and only if G^S is solvable.*

Proposition 2.9.2 makes the notions of regular 1-psg and regular tori compatible: $\lambda(\mathbf{G}_m)$ is a regular torus, if and only if $\lambda(\mathbf{G}_m) \subset T_\alpha$ for no α , if and only if $\lambda \in Y(T)_{\text{reg}}$.

For $\alpha \in \Psi$, put $Z_\alpha = C_G(T_\alpha)$. These later turn out to be relatively simple objects that we can understand well. They play roles analogous to $\mathfrak{sl}_2$ in Lie algebras.

Proposition 2.9.4. Z_α is a group of semisimple rank 1.

Proof. This amounts to proving that $Z_\alpha/R(Z_\alpha)$ has rank 1. But $T_\alpha \subset Z(Z_\alpha)^\circ \subset R(Z_\alpha)$, so $\text{rank}(Z_\alpha/R(Z_\alpha)) = 1$. $\square$

2.9.2 Groups of semisimple rank 1

Theorem 2.9.5. [Hum75, Theorem 25.3] G has semisimple rank 1 if and only if any of the following:

- $|\mathfrak{B}^T| = |W| = 2$
- $\dim G/B = 1$
- $G/B \cong \mathbb{P}^1$
- There exists a surjective morphism $\varphi: G \rightarrow \text{PGL}_2$ such that $\ker \varphi = \bigcap \mathfrak{B}$, so that $(\ker \varphi)^\circ = R(G)$.

Later, we want to apply this to Z_α .

2.9.3 Reductive groups

Suppose now G is reductive.

The key technical theorem is

Theorem 2.9.6.

$$I(T)_u = R_u(G).$$

Proof. Since $R(G) = (\bigcap_{B \in \mathfrak{B}} B)^\circ$ (Proposition 2.7.10), it is clear that $R_u(G)$ is contained in every B_u , and so contained in $I(T)_u$. The main work is in the reverse inclusion. See [Bor91, Theorem 13.16] or [Hum75, Theorem 26.1]. $\square$

The following corollary essentially shows that $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, as we witness later. The importance of the previous theorem is meditated through the next corollary, whose significance we explain in the remark afterwards.

Corollary 2.9.7. If G reductive, S subtorus of T .

1. G^S is reductive.
2. If S regular, $G^S = T$.
3. $G^T = T$.

Proof. To prove (1), we use the above theorem twice: $I(T)_u = R_u(G) = \{e\}$ by G reductive, whereas $R_u(G^S) = I_{G^S}(T)_u$ is the unipotent part of the intersection of all Borel subgroups of G^S containing G^S . But each of such Borel subgroup occurs as $B \cap G^S$ for some Borel subgroup of G containing T (see [Hum75, Corollary 22.4]). Hence $I_{G^S}(T)_u \subset I(T)_u$, and $R_u(G^S) = \{e\}$.

For (2), recall that by Lemma 2.9.3, S being regular implies that G^S is solvable. G^S is also connected, being the centraliser of a torus (see [Hum75, Theorem 22.3]), so the structure theorem of connected solvable groups (Theorem 2.4.1) applies, giving

$$G^S = T \cdot (G^S)_u = T \cdot R_u(G^S) = T,$$

noting that $R(G^S) = G^S$ as G^S is connected and solvable. This proves (2), and (3) follows since maximal tori are regular. $\square$

Remark. Part (3) of this corollary essentially shows $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, while part (1) allows us to apply our understanding of reductive groups of semisimple rank 1 to $Z_\alpha \subset G$. ∇

Another corollary satisfactorily resolves the confusion of having both Φ and Ψ in play.

Corollary 2.9.8. *Let G be reductive. Then $I(T) = T$. Hence $\Psi = \Phi$.*

Proof. Recall that $I(T)$ is the identity component of the intersection of all $B \in \mathfrak{B}^T$. In particular, it is connected, and contains T as a maximal torus. Hence again by the structure theorem of connected solvable groups (Theorem 2.4.1),

$$I(T) = T \cdot I(T)_u = T \cdot R_u(G) = T.$$

It follows from their definitions that $\Psi = \Phi$. $\square$

Combining previous results, we now have

Theorem 2.9.9. *Let G be reductive, and T a maximal torus. Let Φ and Ψ be defined as earlier. Then $\Phi = \Psi$, and*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Proof. We have seen that $\Phi = \Psi$. It remains to see that $\mathfrak{g}^T = \mathfrak{t}$.

Firstly, by the correspondence between global and infinitesimal centralisers (see [Hum75, Section 18.4]), $\mathfrak{g}^T = \mathcal{L}(G^T)$ (this holds without the reductive assumption). But as G is reductive, $G^T = T$ by Corollary 2.9.7 above. So $\mathfrak{g}^T = \mathcal{L}(T) = \mathfrak{t}$. $\square$

We will refine this result, but let us first understand reductive groups of semisimple rank 1 better.

Reductive groups of semisimple rank 1

Let G be reductive of semisimple rank 1. Fix a maximal torus T . By the above, we know $\mathfrak{B}^T$ has size 2, say $\mathfrak{B}^T = \{B, B^-\}$. Put I as the intersection of all Borel subgroups. Denote the Lie algebras as follows:

$$\mathfrak{g} = \mathcal{L}(G), \quad \mathfrak{t} = \mathcal{L}(T), \quad \mathfrak{b} = \mathcal{L}(B), \quad \mathfrak{b}^- = \mathcal{L}(B^-).$$

Proposition 2.9.10. 1. $B_u \cong \mathbf{G}_a \cong B_u^-$, and the actions of T on B_u, B_u^- by conjugation are respectively given by $\alpha, -\alpha$.

2. Regarding $\pm\alpha$ as elements of $X(T)$, we have $\Phi(G, T) = \{\pm\alpha\}, \Phi(B, T) = \{\alpha\}, \Phi(B^-, T) = \{-\alpha\}$

3. $\dim \mathfrak{g}_{\pm\alpha} = 1$, and there are the following equalities of Lie algebras:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \\ \mathfrak{b} &= \mathfrak{t} \oplus \mathfrak{g}_\alpha, \\ \mathfrak{b}^- &= \mathfrak{t} \oplus \mathfrak{g}_{-\alpha}.\end{aligned}$$

Proof. By the results on groups of semisimple rank 1 (Theorem 2.9.5), there is a surjective morphism $\varphi: G \rightarrow \mathrm{PGL}_2$ with $\ker \varphi = I$. Restricting $\varphi: B_u \rightarrow \varphi(B_u) \cong \mathbf{G}_a$, the kernel is $I \cap B_u$. Its identity component is $(I \cap B_u)^\circ \subset I^\circ \cap B_u = R(G) \cap B_u \subset R_u(G) = \{e\}$, hence equality. Since identity components have finite indices, the kernel $I \cap B_u$ is finite.

This implies B_u is connected, unipotent, and one-dimensional. By the classification of one-dimensional connected groups (see [Bor91, section 10.9]), there is an isomorphism $\theta: \mathbf{G}_a \rightarrow B_u$, and moreover T acts by some $\alpha \in X(T/(T \cap I))$ in the sense that

$$t\theta(x)t^{-1} = \theta(t^\alpha x).$$

Passing to PGL_2 , we see that α is a generator, and the action on B_u^- is given by $-\alpha$. This proves (1).

We have $B = T \cdot B_u$, so $\mathfrak{b} = \mathfrak{t} \oplus \mathcal{L}(B_u)$. As $B_u \cong \mathbf{G}_a$ is of dimension 1, so $\mathcal{L}(B_u)$ is one-dimensional. Moreover, $\mathcal{L}(B_u)$ lies inside $\mathfrak{g}_\alpha$. Similarly, $\mathfrak{b}^- = \mathfrak{t} \oplus \mathcal{L}(B^-)$, with $\mathcal{L}(B^-)$ being a one-dimensional subspace of $\mathfrak{g}_{-\alpha}$.

This shows $\mathfrak{b} \cap \mathfrak{b}^- = \mathfrak{t}$. Now $\mathfrak{b} + \mathfrak{b}^- \subset \mathfrak{g}$. We can conclude equality by counting dimensions:

$\dim(\mathfrak{b} + \mathfrak{b}^-) = 2 + \dim T = 3 + \dim I$, since $\dim T = \dim I + \dim \varphi(T) = \dim I + 1$. On the other hand, $\dim \mathfrak{g} = \dim G = \dim I + \dim \mathrm{PGL}_2 = \dim I + 3$. Hence $\mathfrak{b} + \mathfrak{b}^- = \mathfrak{g}$.

This forces that $\mathcal{L}(B_u) = \mathfrak{g}_\alpha, \mathcal{L}(B_u^-) = \mathfrak{g}_{-\alpha}$, and that G has no other roots, i.e. $\Phi(G, T) = \{\pm\alpha\}, \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. $\square$

Proposition 2.9.11.

$$\begin{aligned}\mathfrak{C}(B) &= \{\lambda \in Y(T)_{reg} \mid \langle \alpha, \lambda \rangle > 0\}, \\ \mathfrak{C}(B^-) &= \{\lambda \in Y(T)_{reg} \mid \langle \alpha, \lambda \rangle < 0\}.\end{aligned}$$

Proof. First suppose λ is such that $\langle \alpha, \lambda \rangle > 0$, we will show $B(\lambda) = B$.

We need to recall the construction in Proposition 2.8.6. Using the uniqueness statement there, it suffices to show that there is a open neighborhood U of eB such that for all $x \in U$, we have $\lambda(\infty) \cdot x = eB$.

Let $\theta^- : \mathbf{G}_a \rightarrow B_u^-$ be the isomorphism from the previous proposition. Since B is the stabiliser of eB and $B_u^- \cap B = \{e\}$ and $G/B \cong \mathbb{P}^1$, we know that $\theta^-(a) \cdot (eB)$ covers a open neighborhood of eB . Now, for a fixed a ,

$$\begin{aligned} \lambda(t) \cdot (\theta^-(a) \cdot eB) &= (\lambda(t)\theta^-(a)\lambda(t)^{-1}) \cdot eB \\ &= \theta^-(t^{\langle -\alpha, \lambda \rangle} a) \cdot eB \end{aligned}$$

As $t \rightarrow \infty$, we obtain $\lambda(\infty) \cdot (\theta^-(a) \cdot eB) = \theta^-(0) \cdot eB = eB$. Hence $B(\lambda) = B$. Similarly, when $\langle \alpha, \lambda \rangle < 0$, we have $B(\lambda) = B^-$. This finishes the proof. $\square$

Back to reductive groups in general

Now return to the general setting. Let G be a reductive group and fix a choice of T and B .

Theorem 2.9.12. *Let $\alpha \in \Phi$, then $-\alpha \in \Phi$. Let $Z_\alpha = C_G(T_\alpha)$ as before. Then Z_α is reductive of semisimple rank 1, and $\mathcal{L}(Z_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. In particular, $\dim \mathfrak{g}_\alpha = 1$.*

Proof. We have seen that Z_α is of semisimple rank 1 (Proposition 2.9.4), and it is reductive by Corollary 2.9.7.(1).

Hence Proposition 2.9.10 tells us that it has two roots negative of each other. We now have to identify them with roots in Φ .

But consider its Lie algebra $\mathcal{L}(Z_\alpha) = \mathcal{L}(G^{T_\alpha}) = \mathfrak{g}^{T_\alpha}$ again by the correspondence between global and infinitesimal centralisers ([Hum75, Section 18.4]). Unpacking this, we see that a root space $\mathfrak{g}_\beta$ is contained in $\mathcal{L}(Z_\alpha)$ if and only if $\text{Ad}(t)\mathfrak{g}_\beta = \beta(t)\mathfrak{g}_\beta = 0$ for all $t \in T_\alpha$, i.e. $T_\alpha \subset \ker \beta$.

In particular, α is such a β , and $\mathfrak{g}_\alpha \subset \mathcal{L}(Z_\alpha)$. Now T is a maximal torus of Z_α . So by Proposition 2.9.10, $\Phi(T, Z_\alpha) = \{\pm\alpha\}$, and $\mathcal{L}(Z_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ with $\dim \mathfrak{g}_\alpha = 1$. $\square$

Proposition 2.9.10 further tells us the following: one of the Borel subgroups of Z_α has α as its only root. Denoting its unipotent part by U_α , we have $\mathcal{L}(U_\alpha) = \mathfrak{g}_\alpha$.

Definition 2.9.13. U_α is called the **root subgroup corresponding to α** .

Remark. In fact, U_α is the unique T -stable connected subgroup of G with Lie algebra $\mathfrak{g}_\alpha$. See [Hum75, Section 26.3] ∇

Finally, we examine the important (and illuminating!) **relation between Borel subgroups and roots**.

Theorem 2.9.14. *Let B be a Borel subgroup of G containing T . Write $\Phi(B)$ for $\Phi(T, B)$. Then*

1. Φ is the disjoint union of $\Phi(B)$ and $-\Phi(B)$.
2. $B = B(\lambda)$ if and only if $\langle \beta, \lambda \rangle > 0$ for all $\beta \in \Phi(B)$.

$$3. \mathfrak{b} = \mathcal{L}(B) = \mathfrak{t} \oplus \bigoplus_{\beta \in \Phi(B)} \mathfrak{g}_\beta.$$

Proof. To prove (1), it suffices to show that for every $\alpha \in \Phi$, $\Phi(B)$ intersects $\{\pm\alpha\}$ at precisely one element. Now $B \cap Z_\alpha$ is a Borel subgroup of Z_α , and $\Phi(T, Z_\alpha) = \{\pm\alpha\}$. Since $\Phi(B) \cap \Phi(T, Z_\alpha) = \Phi(B \cap Z_\alpha)$, the result follows from Proposition 2.9.10.

For (2), first assume $B = B(\lambda)$. To determine the roots in $\Phi(B)$, as in (1) consider $\Phi(B \cap Z_\alpha) = \Phi(B) \cap \Phi(T, Z_\alpha)$ for every $\alpha \in \Phi$. By Proposition 2.9.11, we have the following dichotomy: if $\Phi(B \cap Z_\alpha) = \{\alpha\}$, then $\langle \alpha, \lambda \rangle > 0$; if $\Phi(B \cap Z_\alpha) = \{-\alpha\}$, then $\langle \alpha, \lambda \rangle < 0$. Either way, λ pairs positively with the element in $\Phi(B \cap Z_\alpha)$, proving the "only if" part.

Conversely, suppose B is given and $\lambda \in Y(T)_{reg}$ is such that $\langle \beta, \lambda \rangle > 0$ for all $\beta \in \Phi(B)$. Then again by Proposition 2.9.11, we have $B \cap Z_\alpha = B(\lambda) \cap Z_\alpha$ for all $\alpha \in \Phi$. But B (resp. $B(\lambda)$) is generated by these intersections (see [Bor91, Corollary 9.5.(2)]). Hence $B = B(\lambda)$.

(3) is immediate from the definition of $\Phi(B)$. $\square$

Corollary 2.9.15. *Let $B = B(\lambda)$. Then $B^- = B(-\lambda)$ is the unique Borel subgroup such that $\Phi(B^-) = -\Phi(B)$. In this case, $B \cap B^- = T$, and $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{b} \oplus \mathfrak{b}^-$.*

Proof. It is immediate from (2) above that $B(-\lambda)$ is such a group and indeed the unique one. Then (3) above implies the desired equality of Lie algebras.

Now since $T \subset B \cap B^-$, we have $\mathfrak{t} \subset \mathcal{L}(B \cap B^-) \subset \mathfrak{b} \cap \mathfrak{b}^-$. But this is forced to be equality because $\mathfrak{b} \cap \mathfrak{b}^- = \mathfrak{t}$. To conclude from here that $B \cap B^- = T$, see [Bor91, Theorem 14.1, (i) $\implies$ (I)]. $\square$

Definition 2.9.16. B^- is called the Borel subgroup opposite to B (in $\mathfrak{B}^T$). We often write the unipotent parts of B and B^- as U and U^- , respectively. So $B = TU$, $B^- = TU^-$.

As anticipated in the last section, we now know concretely that each Borel subgroup splits the roots Φ into two halves, partitioned by the hyperplane defined by λ if $B = B(\lambda)$. The roots in $\Phi(B)$ are called the **positive roots**. With more work, one can show that Φ is indeed a root system. Then a choice of B specifies a **root basis**. We will not pursue this here, pointing the reader to [Hum75] or [Bor91] for further developments.

Chapter 3

Interlude: Indulgence in Infinitesimal Imprecision

Let G act on a variety X , both over $\mathbb{C}$, inducing a G -action on the function field $\mathbb{C}(X)$ of X by $(g \cdot f)(x) = f(g^{-1} \cdot x)$. The objective of this chapter is to make precise the following: the induced $\mathfrak{g}$ -action is differentiation. We also describe how to transition from G -modules to $\mathfrak{g}$ -modules. This supplements the many details omitted in [Kav15].

Some materials on Lie groups and manifolds are adapted from [FH04] and [Spi99]. Proving various equivalences is my own work.

3.1 G -modules and $\mathfrak{g}$ -modules

Let G be an algebraic group over $\mathbf{k}$, and $\mathfrak{g} = \mathcal{L}(G)$ its Lie algebra. Given a finite-dimensional G -module V , i.e. a morphism of algebraic groups $\rho: G \rightarrow \mathrm{GL}(V)$, there is a natural way to produce a $\mathfrak{g}$ -module structure on V by taking the differential of ρ at e to get $d\rho_e: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. It can be verified that this is a morphism of Lie algebras and hence makes V into a $\mathfrak{g}$ -module. From this definition we know that this behaves suitably functorially: for example, if H is a closed subgroup of G , the two ways of making V into a $\mathfrak{h}$ -module coincide.

On the other hand, let us consider the case when G is now a Lie group and $\rho: G \rightarrow \mathrm{GL}(V)$ a representation of G . The same construction as above makes V into a $\mathfrak{g}$ -module. But in this analytic setting, there is an alternative way to go about this, by using the exponential map $\exp: \mathfrak{g} \rightarrow G$, which we explain now and can be found in more details in [FH04, Lecture 8].

Recall that $\mathfrak{g} = T_e G$ is the tangent space of G at the identity, so any $\xi \in \mathfrak{g}$ can be viewed as a tangent vector at e . Then ξ produces a (left-invariant) vector field $\mathcal{V}_\xi$ on G by translating ξ : formally, let $m_g: G \rightarrow G, h \mapsto gh$ be left multiplication by g , and set $\mathcal{V}_\xi(g) = (m_g)_* \xi$.

Essentially by Picard's theorem in differential equations, given a vector field on a manifold M , we can trace out a curve along any vector field starting at

any $p \in M$. In our case, this produces a unique map $\varphi_\xi: I \rightarrow G$ for some open interval I containing 0 such that $\varphi'_\xi(t) = \mathcal{V}_\xi(\varphi(t))$. In particular, notice that $\varphi'_\xi(0) = \xi$.

Then we define $\exp: \mathfrak{g} \rightarrow G$ by sending ξ to $\varphi_\xi(1)$. Notice that $\exp(s\xi) = \varphi_{s\xi}(1)$ equals $\varphi_\xi(s)$. To see this, simply put $\psi(t) = \varphi_\xi(st)$ and differentiate $\psi(t)$ to see it satisfies the defining properties of $\varphi_{s\xi}(t)$.

The exponential map allows us to define a map $\tilde{\rho}: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ by

$$\xi \rightarrow \frac{d}{dt} \rho(\exp(t\xi))|_{t=0}.$$

We have an equivalence of definitions:

Proposition 3.1.1.

$$\tilde{\rho} = d\rho_e.$$

Proof.

$$\begin{aligned} \tilde{\rho}(\xi) &= d\rho_e\left(\frac{d}{dt} \exp(t\xi)|_{t=0}\right) \\ &= d\rho_e(\varphi'_\xi(0)) = d\rho_e(\xi). \end{aligned}$$

□

In the subsequent sections, we mainly work with algebraic groups over $\mathbb{C}$. But the author of [Kav15] sometimes treats such groups as Lie groups over $\mathbb{C}$ and abuses the language just introduced. We shall follow suit.

The above alternative definition of $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is useful when V is infinite-dimensional. For example, when G acts on a variety M , it makes $\mathbb{C}(X)$ into a (generally infinite-dimensional) G -module by translation of functions. Denote the representation by $\rho: G \rightarrow \mathrm{GL}(\mathbb{C}(X))$. To equip $\mathbb{C}(X)$ with a $\mathfrak{g}$ -action, it is helpful to consider $\tilde{\rho}$ to avoid worrying about tangent spaces of the infinite-dimensional $\mathrm{GL}(\mathbb{C}(X))$.

3.2 G -actions and vector fields on varieties

Let G be a Lie group acting smoothly on a manifold M , with the group action given by $\sigma: G \times M \rightarrow M$.

Given an element $\xi \in \mathfrak{g}$ of the Lie algebra $\mathfrak{g} = \mathcal{L}(G)$, we can associate a vector field $\mathcal{V}_\xi$ on M . We will sometimes call this the **vector field generated by ξ** .

We now describe an abstract construction, given in [Spi99, Chapter 8]. Let $p \in M$ be a point on M . Define $\sigma_p: G \rightarrow M$ by $\sigma_p(g) = \sigma(g, p)$. This is a smooth map between manifolds, mapping $e \in G$ to $p \in M$. Hence we have its differential $(d\sigma_p)_e: \mathfrak{g} \rightarrow T_p M$. Define the vector field $\mathcal{V}_\xi$ by

$$\mathcal{V}_\xi(p) = (d\sigma_p)_e(\xi).$$

Using the exponential map introduced in the last subsection, we can describe $\mathcal{V}_\xi$ more explicitly:

$$\mathcal{V}_\xi(p) = \frac{d}{dt}(\exp(t\xi) \cdot p)|_{t=0} = \frac{d}{dt}\sigma_p(\exp(t\xi))|_{t=0}.$$

The two characterisations agree by a calculation similar to the proof of Proposition 3.1.1 above.

Now suppose G is an algebraic group acting on a variety X , with both G and X over $\mathbb{C}$. We continue to abuse the language from above.

The G -action on X induces a G -action on the function field $\mathbb{C}(X)$ of X by $(g \cdot f)(x) = f(g^{-1} \cdot x)$. Using the results of the previous subsection (and by abuse of language), we have a $\mathfrak{g}$ -module structure on $\mathbb{C}(X)$. Since $\mathbb{C}(X)$ is generally infinite-dimensional, we need to use the alternative formulation. Explicitly, we have

$$(\xi \cdot f)(x) = \frac{d}{dt}f(\exp(t\xi)^{-1} \cdot x)|_{t=0}.$$

Let us now consider how this relates to the vector field $\mathcal{V}_\xi$ on X . Assume in addition that G acts on X *without fixed point*, i.e. that if $g \cdot x = x$ for *some* x , then $g = e$. Equivalently, any $g \neq e$ acts non-trivially at any point $x \in X$. In particular, this implies that $\mathcal{V}_\xi$ is nowhere vanishing, and one can take the derivative of f along $\mathcal{V}_\xi$. Then **the point of all this** is that we have the following

Proposition 3.2.1. *Suppose G acts on X without fixed point. Let $\xi \in \mathfrak{g}$ and $f \in \mathbb{C}(X)$. Then $\xi \cdot f$ equals the derivative of f along $\mathcal{V}_{-\xi}$:*

$$(\xi \cdot f)(x) = df(\mathcal{V}_{-\xi}(x)).$$

Proof. We combine the descriptions of $\xi \cdot f$ and $\mathcal{V}_{-\xi}$ involving $\exp$.

$$\begin{aligned} (\xi \cdot f)(x) &= \frac{d}{dt}f(\exp(t\xi)^{-1} \cdot x)|_{t=0} \\ &= df\left(\frac{d}{dt}\exp(-t\xi) \cdot x|_{t=0}\right) \quad [\text{chain rule}] \\ &= df(\mathcal{V}_{-\xi}(x)) \quad [\text{exp-definition of } \mathcal{V}_{-\xi}]. \end{aligned}$$

□

Therefore, we know that $\mathfrak{g}$ -actions on rational functions on X are differentiation. Consequently, we have a Leibniz rule:

Corollary 3.2.2 (Leibniz rule). *Let $\xi \in \mathfrak{g}$ and $f, g \in \mathbb{C}(X)$. Then*

$$\xi \cdot (fg) = f(\xi \cdot g) + (\xi \cdot f)g.$$

Proof. This follows from the proposition above and $d(fg) = f dg + g df$. □

Remark. The statement of Proposition 3.2.1 makes perfect sense in the world of varieties, so we make some brief remarks on the possibility of its proof without abusing the as we did. Firstly, perhaps one could try to directly relate the intrinsic definitions of the $\mathfrak{g}$ -action and of the vector field without using the “1-parameter” technique afforded by the exponential map, but I’m not sure how to do this.

A more promising approach may be to start with an algebraic reformulation of the vector field $\mathcal{V}_\xi$. When G is an (affine) algebraic group over $\mathbf{k}$, derivations of the ring $\mathbf{k}[G]$ give a good notion of vector fields on G . Then when G acts on itself by left translations, giving for any $X \in \mathfrak{g}$ a left-invariant vector field on G , we can capture this vector field by a left-invariant derivation. Explicitly, $*X$ (right convolution by X) produces a left-invariant derivation of G by $(f * X)(x) = X(\lambda_{x^{-1}}f)$. We may try to generalise this to arbitrary varieties X by considering the left-invariant derivations of $\mathbf{k}[X]$ or $\mathbf{k}(X)$. The problem is that with considerably less knowledge of the G -action X , it is not clear how to generalise the notion of convolutions. We may need some constraints on the G -action for this to work, for example by assuming it acts without fixed point. ∇

Chapter 4

Polynomial Expressions for Irreducible Representations

From this point onwards, G will always be a connected semisimple algebraic group over $\mathbb{C}$, unless otherwise noted. Let $\mathfrak{g} = \mathcal{L}(G)$. In this chapter, we begin the task of obtaining geometric interpretations for the irreducible representations of G .

In the first section, we introduce the remarkable theorem of Borel-Weil-Bott, which states that any irreducible G -module can be realised as the global sections of a G -equivariant line bundle over the flag variety G/B such that the induced $\mathfrak{g}$ -actions correspond to *differentiation of sections*. Although we do not prove the theorem, we give the construction of the line bundles and explicitly describe the G -actions on their global sections. This latter part is not easily found in the literature.

Modulo the notation properly given in the next section, this gives a line bundle L_λ for each dominant integral weight λ , whose global sections $\Gamma(G/B, L_\lambda)$ are isomorphic to V_λ^* , the unique irreducible representation of lowest weight $-\lambda$.

As mentioned in the Introduction, familiar examples of line bundles in algebraic geometry motivate us to seek a way of expressing global sections of $\Gamma(G/B, L_\lambda)$ also as polynomials. The overview of this development will be given in Section 4.2, after we introduce the Borel-Weil-Bott Theorem. The polynomial expressions we obtain not only give us a more concrete grasp on V_λ^* , but also provide some interesting combinatorial data associated to the irreducible representations.

Independently, the theory of crystal bases also provides combinatorial descriptions for the irreducible representations of G . We will then explore a beautiful connection between these two combinatorial views in Chapter 5.

4.1 Irreducible representations as line bundles

L_λ : the Borel-Weil-Bott Theorem

Let G be a connected semisimple algebraic group over $\mathbb{C}$, and let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be its Lie algebra. Fix a maximal torus T and a Borel subgroup B containing T . In particular, B specifies a set of positive roots and a root basis Δ consisting of simple roots. **Let these choices be fixed for the remainder of the essay.**

We know that finite-dimensional irreducible G -modules are in 1-to-1 correspondence with the dominant weights of G ([Hum75, Section 31]). If $\lambda \in X(T)$ is a dominant weight (i.e. pairing positively with all positive roots), let V_λ denote the unique finite-dimensional simple G -module of highest weight λ .

Now consider V_λ as a $\mathfrak{g}$ -module. It is irreducible because we are in characteristic 0 ([Hum75, Theorem 13.2]), so it has to be isomorphic to the irreducible $\mathfrak{g}$ -module $V_{d\lambda}$, where $d\lambda \in \mathfrak{t}^*$ is the differential of $\lambda: T \rightarrow \mathbf{G}_m$. Indeed, one knows that $d\lambda$ must be an *integral* dominant weight ([Ma]). When we consider V_λ as a $\mathfrak{g}$ -module, we will identify $d\lambda$ with λ , and also say it has highest weight λ .

The following remarkable geometric construction that realises V_λ as the global sections of a line bundle over the flag variety G/B . This is part of the content of the Borel-Weil-Bott theorem (the full statement also concerns higher cohomologies, not just global sections, but we will not need those).

Theorem 4.1.1 (Borel-Weil-Bott Theorem). *Let G and B be as above. For every integral weight λ , there exists a G -equivariant line bundle L_λ on G/B . Its space of global sections $\Gamma(G/B, L_\lambda)$ is a $\mathfrak{g}$ -representation, with $\mathfrak{g}$ acting by differentiation. If λ is dominant, then $\Gamma(G/B, L_\lambda) \cong V_\lambda^*$ is the dual of V_λ – equivalently, the unique simple module of lowest weight $-\lambda$.*

We will not give the proof, but we will describe the construction of L_λ and the actions of G and $\mathfrak{g}$. The full proof can be found in, for instance, [Ma] or [Lur].

It is known that any character σ of T extends to a character $\sigma: B \rightarrow \mathbf{G}_m$ [Ma]. So from λ we obtain a character $\sigma_\lambda: B \rightarrow \mathbf{G}_m = \mathrm{GL}(\mathbb{C})$, i.e. a one-dimensional representation of B .

We next describe how to construct the G -equivariant line bundle L_λ out of this data, using the general construction of associated fibre bundles.

Given a subgroup H of G and an H -action on a variety X , define

$$G \times_H X = G \times X / (g, x) \sim (gh^{-1}, h \cdot x).$$

[The general theory of associated fibre bundles states that $G \times_H X$ is a well-defined variety whenever $G \rightarrow G/H$ has local sections. Moreover, it comes with a projection morphism $G \times_H X \rightarrow G/H$ with local sections. For details, see [Spr98, Lemma 5.5.8].]

When $G \times_H X$ is well-defined, we call it the **associated fibre bundle** $G \times_H X$ **over** G/H **with fibre** X .

In our case, let $\sigma_\lambda: B \rightarrow \mathrm{GL}(\mathbb{C})$ be as above. Define $L_\lambda = G \times_B \mathbb{C}$, i.e.

$$L_\lambda = G \times \mathbb{C} / (g, v) \sim (gb^{-1}, \sigma_\lambda(b)v).$$

Then L_λ is well-defined since $G \rightarrow G/B$ has local sections (this is proved with intricate manipulations with the Bruhat decomposition; see [Spr98, Lemma 8.5.2]), and it comes with a projection map $\pi: G \times_B \mathbb{C} \rightarrow G/B$. Also, note that L_λ has a G -action by left multiplication on the first coordinate.

Explicitly, $\pi(g, v) = gB$. One can then easily check π is G -equivariant and that $\pi^{-1}(gB)$ has a vector space structure. For example, $\alpha \cdot [(g, v)] := [(g, \alpha v)]$ is well-defined: $\alpha \cdot [(gb^{-1}, b \cdot v)] = [(gb^{-1}, \alpha(b \cdot v))] = [(gb^{-1}, b \cdot (\alpha v))] = [(g, \alpha v)] = \alpha \cdot [(g, v)]$.

A global section of L_λ is a morphism $s: G/B \rightarrow L_\lambda$ such that $\pi \circ s = \mathrm{id}_{G/B}$. Then from the equivalence relation above, we see that a section s gives a unique $f \in \mathbb{C}[G]$ such that $\sigma(gB) = [(g, f(g))]$ and

$$f(gb^{-1}) = \sigma_\lambda(b)f(g)$$

(note that each fibre is simply $\mathbb{C}$). Conversely, any such f determines a section s .

So we can identify $\Gamma(G/B, L_\lambda)$ with

$$\Gamma(\lambda) := \{f \in \mathbb{C}[G] \mid f(gb^{-1}) = \sigma_\lambda(b)f(g)\}.$$

We can then define a G -action on $\Gamma(G/B, L_\lambda)$ by defining the corresponding G -action on $\Gamma(\lambda)$. Let this action be the usual translation of functions: if $f \in \Gamma(\lambda)$, define $(g \cdot f)(h) = f(g^{-1}h)$. It is clear that $\Gamma(\lambda)$ is stable under this action.

Hence $\Gamma(G/B, L_\lambda)$ acquires the structure of a G -module. This then induces **its structure as a $\mathfrak{g}$ -module** (see Section 3.1). The $\mathfrak{g}$ -action is **differentiation of sections** in the sense that it is the differentiation of their representing functions (as in Proposition 3.2.1).

Let us (me!) also observe how the quotient of two global sections explicitly gives a rational function on G/B . Let $f_1, f_2 \in \Gamma(\lambda)$. Then

$$(f_1/f_2)(gb^{-1}) = f_1(g)/f_2(g)$$

since the coefficients $\sigma_\lambda(b)$ cancel out. This descends to a rational function on G/B .

Hence by choosing a non-zero section τ , we can define an embedding $\Gamma(G/B, L_\lambda) \rightarrow \mathbb{C}(G/B)$ by sending $\sigma \mapsto f_\sigma = \sigma/\tau$.

4.2 $\Gamma(G/B, L_\lambda)$ as polynomials: the road map

It is already striking that the Borel-Weil-Bott Theorem gives a geometric realisation of the $\mathfrak{g}$ -module V_λ^* as $\Gamma(G/B, L_\lambda)$ such that the $\mathfrak{g}$ -actions correspond to differentiation of sections.

To express the sections as polynomials, one natural place to start is to embed $\Gamma(G/B, L_\lambda)$ into the function field $\mathbb{C}(G/B)$ of the underlying variety G/B . This can be done by choosing any non-zero section $\tau \in \Gamma(G/B, L_\lambda)$, and defining $\Gamma(G/B, L_\lambda) \rightarrow \mathbb{C}(G/B)$ by sending $\sigma \mapsto f_\sigma = \sigma/\tau$. For our purposes, there is a good choice of τ we can make, such that

1. This embedding preserves “half of” the $\mathfrak{g}$ -module structure on V_λ^* and makes it compatible with the one on $\mathbb{C}(G/B)$, to be introduced later.
2. f_σ becomes regular (i.e. has no poles) over an open subset $\mathcal{U}^- \subset G/B$ (the open opposite cell, after we introduce the Bruhat decomposition).

Once we made this good choice of τ , we will introduce an auxiliary variety $\tilde{X}_{w_0}$ (this notation is justified in the next section), called a Bott-Samelson variety, which is equipped with a birational morphism to G/B . Recall that birational morphisms allow us to identify function fields of varieties.

Definition 4.2.1. A morphism of varieties $\varphi: X \rightarrow Y$ is a **birational morphism** if the induced map (by pre-composition with φ) on function fields $\varphi^*: \mathbb{C}(Y) \rightarrow \mathbb{C}(X)$ is an isomorphism.

Remark. Note that we have *not* defined a birational *map*. A birational morphism is really a morphism, defined on all of X . ∇

The crucial property of $\tilde{X}_{w_0}$ of use to us is that it contains an affine open set $\tilde{\mathcal{U}}$ isomorphic to some $\mathbb{C}^N$ mapping to $\mathcal{U}^-$. This then allows us to identify the regular function $f_\sigma \in \mathbb{C}[U^-]$ with one on $\mathbb{C}^N$, which then is a polynomial.

In the next section, we first discuss the choice of τ and the issue of preserving $\mathfrak{g}$ -module structures. Then, we introduce the part of Schubert varieties and Bott-Samelson varieties that is needed for expressing f_σ as a polynomial. Combining these two, we obtain the desired conclusion.

4.3 Embedding into $\mathbb{C}(G/B)$ and what is preserved

In Section 4.1, we have described the $\mathfrak{g}$ -module structure of $\Gamma(G/B, L_\lambda)$, which is by *differentiation of sections*. Moreover, there we pointed out choosing a non-zero section $\tau \in \Gamma(G/B, L_\lambda)$ defines an embedding $\Gamma(G/B, L_\lambda) \hookrightarrow \mathbb{C}(G/B)$, $\sigma \mapsto f_\sigma = \sigma/\tau$.

On the other hand, since G acts on G/B by left multiplication, we have also a $\mathfrak{g}$ -module structure on $\mathbb{C}(G/B)$ which is by *differentiation of rational functions*, as in Section 3.2.

We now investigate to what extent we can make this embedding respect the $\mathfrak{g}$ -module structures on $\Gamma(G/B, L_\lambda)$ and $\mathbb{C}(G/B)$. The following proposition makes essential use of the theory developed in Section 3.2. It tells us that **(some) differentiation of sections gets carried over to differentiation of functions.**

Proposition 4.3.1. *Let $\tau_\lambda \in V_\lambda^*$ be a lowest weight vector. Then the embedding $\sigma \mapsto f_\sigma = \sigma/\tau_\lambda$ described above via τ_λ respects the action of U^- in the following sense: if α is a positive root, then*

$$F_\alpha \cdot \sigma = (F_\alpha \cdot f_\sigma)\tau_\lambda$$

for all $\sigma \in \Gamma(G/B, L_\lambda)$

Proof. As explained after constructing L_λ (at the end of Section 4.1), we may view σ, τ_λ as elements of $\mathbb{C}[G]$, and f_σ as element of $\mathbb{C}(G)$.

Now U^- acts on G without fixed points, so we can apply the Leibniz rule (Corollary 3.2.2):

$$F_\alpha \cdot \sigma = (F_\alpha \cdot f_\sigma)\tau_\lambda + f_\sigma(F_\alpha \cdot \tau_\lambda).$$

But as τ_λ is a lowest weight vector, $F_\alpha \cdot \tau_\lambda = 0$, and we obtain the desired equality. $\square$

Remark. The statement of this proposition appears in the proof of the main theorem (Theorem 4.1) of [Kav15]. Sections 3 and 4.1 are essentially the effort to fill in the details missing from the author's deductions. This may or may not be the original intended argument. ∇

4.4 Schubert varieties and Bott-Samelson varieties: first encounter

The first two subsections can be largely ignored on the first read if the reader only wishes to understand how to express f_σ as a polynomial. Only the definition of $\mathcal{U}^-$ (given in the short third subsection) and the fact that the Schubert variety X_{w_0} is the whole of G/B (Proposition 4.4.7) are needed for this purpose.

The sources consulted in the preparation of this section include [Ful96], [BK05], [Spr98], [WY23].

4.4.1 Bruhat decomposition

Schubert varieties are indexed by elements of the Weyl group W , so that each $w \in W$ gives rise to a Schubert variety X_w . Schubert varieties are generally singular, and Bott-Samelson varieties are introduced to resolve their singularities, although for us we do not (seem to) use their smoothness in any essential way.

The Schubert variety X_{w_0} corresponding to the long element w_0 deserves special mention; it is in fact the whole of G/B . This is why the Bott-Samelson variety in the last section is denoted $\tilde{X}_{w_0}$.

To define X_w , we need to introduce the Bruhat decomposition.

Let G be a reductive algebraic group, $W = N_G(T)/C_G(T) = N_G(T)/T$ be its Weyl group.

For an element $w \in W$, let $\sigma \in N_G(T)$ be a representative of w . Define $BwB \subset G$ to be the double coset $B\sigma B \subset G$. Since $T \subset B$, this does not depend on the choice of representative σ .

The Bruhat decomposition of G states that

Theorem 4.4.1. G is the disjoint union of BwB , where w ranges over W .

Remark. Observe that it is purely group-theoretical that any two double cosets are either identical or disjoint. So the content of the theorem is that $BWB = G$. ∇

We will not prove this theorem in general, but it is not hard to explain why this is true when $G = \mathrm{GL}_n$.

Example 4.4.2 (Bruhat decomposition in $\mathrm{GL}(n, \mathbf{k})$). Let $G = \mathrm{GL}(n, \mathbf{k})$, and $T = \mathrm{D}(n, \mathbf{k}) \subset G$ a maximal torus, and B the Borel subgroup consisting of upper-triangular matrices.

We will show that given any matrix $M \in G$, by pre-multiplying and post-multiplying elements of B , we can transform it into a permutation matrix. Permutation matrices visibly normalise the diagonal matrices T , so represent elements of W , which shows $M \in BwB$. (In fact, $W = S_n$ in this case by looking at the root data of G , though we don't need this now.)

Now, pre-multiplying M an upper-triangular matrix performs *row* operations in M where each row is replaced by sums of multiples of itself and rows *below* it. It is convenient to think of this as that we can pick a row in M and *sweep up* using this row. Similarly, post-multiplying an upper-triangular corresponds to *sweeping right*.

The row and column operations available this way are precisely the operations we need to put M into both row and column echelon forms. Since M is invertible, **in row and column echelon form M becomes a permutation matrix.**

Since our conventions of echelon forms differ from the usual conventions, we still describe the procedure.

Begin with row operations. This is the same as usual, except that we start from the *bottom* row and work upwards. At each row, select the first non-zero element (if any) on the *left*. This element is sometimes call the pivot of the row. Scale the pivot to 1, and sweep up to reduce all entries above it to 0. Now move to the row above and repeat.

By assumption, all entries to the left of pivots are 0. Moreover, no two pivots are in the same column by construction. So we can use column operations to sweep right and reduce all entries to the right of pivots also to 0 (each time only affecting one row).

Now M is reduced to an invertible matrix with at most one 1 in each row, all in different columns, and zeros elsewhere. By invertibility, there is exactly one 1 each row. $\diamond$

4.4.2 Schubert cells and Schubert varieties

We return to the general setup.

Definition 4.4.3. Given $w \in W$, the **Schubert cell** $C_w \subset G/B$ is the image of BwB in G/B . The Schubert variety X_w is the (Zariski) closure of C_w in G/B .

By Bruhat decomposition, the Schubert cells C_w partition G/B . Since C_w is simply the B -orbit (B acting on the left) of $wB \in G/B$, it is B -stable. Hence $X_w = \overline{C_w}$ is B -stable: if $b \in B$, then

$$b \cdot \overline{C_w} \supset b \cdot C_w = C_w \implies b \cdot \overline{C_w} \supset \overline{C_w},$$

i.e. $b \cdot X_w \supset X_w$. Similarly, $b^{-1} \cdot X_w \supset X_w$, so $b \cdot X_w = X_w$.

In particular, we know that any Schubert variety X_w decomposes as the disjoint union of the B -orbits, i.e. the disjoint union of Schubert cells C_v , for some $v \in W$.

Note that $X_w \supset C_v$ if and only if $X_w \supset X_v$, since $X_v = \overline{C_v}$. This defines a partial order on W .

Definition 4.4.4. The **Bruhat order** on W is defined by declaring $w \geq v$ if $X_w \supset X_v$.

We can describe the Bruhat order in terms of the combinatorics of W . First, a notation:

Notation 4.4.5. When $\underline{w} = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_d})$ is an ordered d -tuple of simple roots, we write $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \dots s_{\alpha_{i_d}}$ for the corresponding product of simple reflections. Conversely, $\underline{w}$ is called a **word** or a **decomposition** of w . It is **reduced** if the length of $\underline{w}$ equals the length $\ell(w)$ of w in W .

Theorem 4.4.6. Let $\underline{w} = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_d})$ be a reduced word for $w \in W$. Then $v \leq w$ if and only if v has a word that is a sub-tuple of $\underline{w}$.

Proof. [Spr98, Proposition 8.5.5]. □

In particular, we get the same v 's with any word $\underline{w}$ for w , since the Bruhat order itself does not depend on words.

It follows from this that

Proposition 4.4.7. $X_{w_0} = G/B$.

4.4.3 Opposite Schubert cells

Similarly one can define opposite Schubert cells. Let B^- be the Borel subgroup opposite to B .

Definition 4.4.8. Given $w \in W$, the opposite Schubert cell C_w^- is defined to be the image of B^-wB in G/B .

We in fact do not need this generality and only need the opposite Schubert cell C_1^- corresponding to the identity $1 \in W$. This is by definition the image of $B^-B \subset G$ in G/B .

Since $B^- = TU^-$ and $T \subset B$, we have $B^-B = U^-B$. This is an open (dense) subset of G (see [Hum75, Proposition 28.5]). Hence so is its image C_1^- in G/B .

We define $\mathcal{U}^- = C_1^-$ and call it the open opposite cell. Note that it evidently contains the identity $eB \in G/B$.

4.4.4 Bott-Samelson varieties

We will now use the theory of Bott-Samelson varieties to construct a variety $\tilde{X}_{\underline{w}_0}$ with a birational morphism $\tilde{X}_{\underline{w}_0} \rightarrow X_{w_0} = G/B$.

Definition 4.4.9. For a simple root α , we write P_α for the **minimal standard parabolic subgroup of α** , which is the subgroup generated by B and a representative $\bar{s}_\alpha \in N_G(T)$ of s_α . This is independent of the choice of the representative $\bar{s}_\alpha$ because the Borel subgroup B contains the centraliser $C_G(T)$ (Proposition 2.8.3).

Using Notation 4.4.5 from last section, let $\underline{w} = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_d})$ be a word. We define the Bott-Samelson variety $\tilde{X}_{\underline{w}}$ as

$$\tilde{X}_{\underline{w}} = (P_{\alpha_{i_1}} \times \dots \times P_{\alpha_{i_d}}) / B^d,$$

where B^d acts on $P_{\alpha_{i_1}} \times \dots \times P_{\alpha_{i_d}}$ from the right via:

$$(p_1, \dots, p_d) \cdot (b_1, \dots, b_d) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{d-1}^{-1} p_d b_d).$$

Note that from the definition of the B^d -action, there is a well-defined map

$$\pi_{\underline{w}}: \tilde{X}_{\underline{w}} \rightarrow G/B, (p_1, \dots, p_N) \pmod{B^d} \mapsto \prod_i p_i \pmod{B}.$$

It is known (for example, from [Bri10]) that $\tilde{X}_{\underline{w}_0}$ is a smooth projective variety of dimension d (we will not explicitly use this fact). Moreover, we have

Theorem 4.4.10. *Suppose that $\underline{w}$ is a reduced word. Then $\pi_{\underline{w}}: \tilde{X}_{\underline{w}} \rightarrow G/B$ is birational onto its image, which is the Schubert variety $\tilde{X}_{\underline{w}}$. Moreover, $\pi_{\underline{w}}$ is B -equivariant, and restricts to an isomorphism over the Schubert cell $\mathbb{C}_w$.*

Now let $w_0 \in W$ be the long element of the Weyl group, and fix a reduced decomposition $\underline{w}_0 = (\alpha_{i_1}, \dots, \alpha_{i_N})$.

The above construction gives us a morphism $\pi_{\underline{w}_0}: \tilde{X}_{\underline{w}_0} \rightarrow G/B$ which is birational onto G/B , as we desired.

Now consider the map $\tilde{\Phi}_{\underline{w}_0}: \mathbb{C}^N \rightarrow U_{\alpha_{i_1}}^- \times \dots \times U_{\alpha_{i_N}}^- \rightarrow X_{\underline{w}_0}$ given by:

$$\tilde{\Phi}_{\underline{w}_0}: (t_1, \dots, t_N) \mapsto (\exp(t_1 F_{\alpha_{i_1}}), \dots, \exp(t_N F_{\alpha_{i_N}})) \pmod{B^N}.$$

Then one has the following

Proposition 4.4.11. $\tilde{\Phi}_{\underline{w}_0}$ is an open embedding whose image is a neighborhood $\tilde{\mathcal{U}}$ of the identity in $\tilde{X}_{\underline{w}_0}$. Moreover, $\tilde{\mathcal{U}}$ maps under $\pi_{\underline{w}_0}$ to the open opposite cell $\mathcal{U}^- \subset G/B$.

Proof. See [Mag98]. □

Therefore, regular functions $\mathbb{C}[\tilde{\mathcal{U}}]$ on $\tilde{\mathcal{U}} \subset \tilde{X}_{\underline{w}_0}$ are identified with regular functions on $\mathbb{C}^N$. Using the embedding $\tilde{\Phi}_{\underline{w}_0}$, we can write such regular functions as polynomials in variables $t_1, \dots, t_n$. We sometimes call $(t_1, \dots, t_n)$ the Bott-Samelson coordinates on $\tilde{\mathcal{U}}$.

4.5 Global sections of L_λ are polynomials

Using the Bott-Samelson coordinates, we can now express elements of the representation $V_\lambda^* \cong \Gamma(G/B, L_\lambda)$ as polynomials, as promised.

Step 1: We will show that under the embedding given in Section 4.3, the image of any section becomes regular when restricted to $\mathcal{U}$.

Let $\sigma \in \Gamma(G/B, L_\lambda)$, and let τ_λ be the lowest weight vector chosen before. Now τ_λ being lowest weight, it is a B^- -eigenvector. Recalling that the G -action on sections are induced translations of functions (Section 4.1), the divisor D_λ of τ_λ is also B^- -invariant. Therefore, D_λ does not meet the opposite open cell $\mathcal{U}^-$. This is because $\mathcal{U}^-$ is by construction the B^- -orbit of eB , so if D_λ meets $\mathcal{U}^-$, we would have D_λ containing the entire $\mathcal{U}^-$, which is a contradiction as $\mathcal{U}^-$ is open and top-dimensional, while D_λ is a (codimension 1) divisor.

In particular, τ_λ does not vanish on $\mathcal{U}^-$, and f has no poles on $\mathcal{U}^-$, i.e. $f \in \mathbb{C}[\mathcal{U}^-] \subset \mathbb{C}(G/B)$ is regular on $\mathcal{U}^-$.

Step 2: Since the birational morphism $\pi_{\underline{w}_0}$ induces isomorphisms of function fields and maps $\tilde{\mathcal{U}}$ to $\mathcal{U}^-$, we can pullback f to a regular function $\tilde{f}$ on $\tilde{\mathcal{U}}$. The final remarks of the last section show that this can be viewed as a polynomial in $t_1, \dots, t_n$.

With polynomial expressions for the representation V_λ^* , we can explore their attendant combinatorics, done in the next chapter.

Chapter 5

Links to String Parametrisation of Crystal Bases

In Section 4.3, it was shown that embedding $\Gamma(G/B, L_\lambda)$ into $\mathbb{C}(G/B)$ preserves “half” of its $\mathfrak{g}$ -module structure. It is now natural to ask to what extent our further passage to expressing the sections as polynomials preserves this structure. For example, does differentiation of functions on $\mathbb{C}(G/B)$ become differentiation of polynomials?

The answer is not exactly yes. As explained below, we lose access to some $\mathfrak{g}$ -actions as we pass to the Bott-Samelson varieties. But as anticipated in the last chapter, there are still enough combinatorial structures for us to make a connection to the combinatorics of crystal bases.

On the one hand, we can define a natural notion of “highest term valuation” on any polynomial ring that essentially records the highest exponent of any given polynomial. For example, in the univariate polynomial ring $\mathbf{k}[x]$, the degree function $\deg: \mathbf{k}[x] \rightarrow \mathbb{Z}_{\geq 0}^n$ is such a function. In the next section, we will generalise this to multivariate rings. Applying this valuation to our expressions $\sigma \in \Gamma(G/B, L_\lambda)$ associates an N -tuple of integers to each σ .

On the other hand, we will introduce the theory of crystal bases of G -representations. For each G -representation V , there is a special basis called the crystal basis satisfying some remarkable combinatorial properties. The procedure of string parametrisation produces an N -tuple for each crystal basis vector, called its string parameters. The string parameters enjoy the remarkable properties that they embed the crystal basis of $\Gamma(G/B, L_\lambda)$ as integral points inside a polytope in the Euclidean space. We will extend the definition of string parametrisation to the entire representation.

The punchline is that string parametrisation and highest term valuations coincide. Indeed, in the proofs we will see that in the passage to polynomial expressions, almost precisely enough information is preserved to go between the

two.

5.1 Valuations on polynomial rings

In the polynomial ring $\mathbf{k}[x]$ in one variable, we can naturally order the monomials x^d by their degrees. We would like to generalise this notion to polynomial rings $\mathbf{k}[x_1, \dots, x_n]$ of many variables.

Identifying $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ with the monomial $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, we define a monomial ordering as follows.

Definition 5.1.1. A monomial ordering $<$ on $\mathbf{k}[x_1, \dots, x_n]$ is a relation $>$ on $\mathbb{Z}_{\geq 0}^n$, satisfying:

- $<$ is a total well-ordering on $\mathbb{Z}_{\geq 0}^n$, i.e. any two distinct element is comparable, and the relation is transitive, and any non-empty subset of $\mathbb{Z}_{\geq 0}^n$ has a minimal element.
- it respects the additive structure on $\mathbb{Z}_{\geq 0}^n$ (hence the multiplicative structure in the ring): if $\alpha < \beta$, then $\alpha + \gamma > \alpha + \gamma$ for all γ .

The **lexicographic ordering** below is easily seen to be an example, and it is also the ordering of interest to us.

Definition 5.1.2. In the lexicographic ordering, $\alpha < \beta$ if the leftmost non-zero entry of $\alpha - \beta$ is negative. One notices that this is analogous to how words in dictionaries are ordered, hence the name “lexicographic”.

Example 5.1.3. In this ordering, $x_1 > x_2 > \cdots > x_n$. $\diamond$

Now it is straightforward to define the highest term valuation on $\mathbf{k}[x_1, \dots, x_n]$.

Definition 5.1.4. Given a lexicographic ordering as above, the highest term valuation is a map $v: \mathbf{k}[x_1, \dots, x_n] \rightarrow \mathbb{Z}_{\geq 0}^n$ such that $v(f) = (\alpha_1, \dots, \alpha_n)$ if (some non-zero multiple of) x^α is the monomial with the highest lexicographic order occurring in f .

In the last chapter, dependent on a choice of reduced word $\underline{w}_0$ for $w_0 \in W$, we have expressed $\Gamma(G/B, L_\lambda)$ as polynomials in $\mathbf{k}[t_1, \dots, t_N]$. Using lexicographic ordering on $\mathbf{k}[t_1, \dots, t_N]$ with $t_1 > t_2 > \cdots > t_N$, define a highest term valuation on $\Gamma(G/B, L_\lambda)$. Denote this by $v_{\underline{w}_0}: \Gamma(G/B, L_\lambda) \rightarrow \mathbb{Z}_{\geq 0}^N$.

5.2 Crystal bases

As always, G is a connected semisimple algebraic group over $\mathbb{C}$, with a choice of T and B fixed. Let V be a finite-dimensional G -module, hence also a $\mathfrak{g}$ -module.

Let α be a simple root with the corresponding root space generators $E_\alpha \in \mathfrak{g}_\alpha$ and $F_\alpha \in \mathfrak{g}_{-\alpha}$. Define the functions $\epsilon_\alpha, \varphi_\alpha : V \setminus \{0\} \rightarrow \mathbb{Z}$ by:

$$\begin{aligned}\epsilon_\alpha(v) &= \max\{a \mid E_\alpha^a \cdot v \neq 0\}, \\ \varphi_\alpha(v) &= \max\{a \mid F_\alpha^a \cdot v \neq 0\}.\end{aligned}$$

If $E_\alpha \cdot v = 0$ (respectively $F_\alpha \cdot v = 0$), we put $\epsilon_\alpha(v) = 0$ (respectively $\varphi_\alpha(v) = 0$). These are well-defined by the finite-dimensionality of V . We think of $\epsilon_\alpha(v)$ and $\varphi_\alpha(v)$ as the maximal number of steps we can move up or down from v along the direction corresponding to α .

The representation V has a special basis $\mathcal{B}_V$, called the **crystal basis**, consisting of weight vectors and satisfying the following properties ([BK07, Lemma 5.50]): For each $b \in \mathcal{B}_V$ let $E_\alpha \cdot b = \sum_i c_i b_i$ with $b_i \in \mathcal{B}_V, c_i \neq 0$. Then

- (i) For every b_i , $\epsilon_\alpha(b_i) \leq \epsilon_\alpha(b) - 1$.
- (ii) If $E_\alpha \cdot b \neq 0$, then there is a unique k with $\epsilon_\alpha(b_k) = \epsilon_\alpha(b) - 1$. For all other $i \neq k$, $\epsilon_\alpha(b_i) < \epsilon_\alpha(b) - 1$.

Similarly, for each $b \in \mathcal{B}_V$ let $F_\alpha \cdot b = \sum_j e_j b_j$, $e_j \neq 0$. Then

- (iii) For every b_j , $\varphi_\alpha(b_j) \leq \varphi_\alpha(b) - 1$.
- (iv) If $F_\alpha \cdot b \neq 0$, there exists a unique ℓ with $\varphi_\alpha(b_\ell) = \varphi_\alpha(b) - 1$. For all other $j \neq \ell$, $\varphi_\alpha(b_j) < \varphi_\alpha(b) - 1$.

In the above, define $\tilde{E}_\alpha(b) = b_k$ and $\tilde{F}_\alpha(b) = b_\ell$. If $E_\alpha(b) = 0$ (respectively $F_\alpha(b) = 0$), let $\tilde{E}_\alpha(b) = 0$ (respectively $\tilde{F}_\alpha(b) = 0$). Then $\mathcal{B}_V$ satisfies additionally that $\tilde{E}_\alpha(b) = b'$ if and only if $\tilde{F}_\alpha(b') = b$ for $b, b' \in \mathcal{B}_V$.

The operators $\tilde{E}_\alpha, \tilde{F}_\alpha : \mathcal{B}_V \rightarrow \mathcal{B}_V \cup \{0\}$ are called the **Kashiwara operators** corresponding to the simple root α .

It is known that the dual basis $\mathcal{B}_\lambda^*$ is also a crystal basis for the dual representation V_λ^* ([BK07, Lemma 5.50] and [Kah+14, Proposition 4.3 and Proposition 7.3]).

Since we have been working with $\Gamma(G/B, L_\lambda) \cong V_\lambda^*$, we will mainly focus on the dual crystal basis $\mathcal{B}_\lambda^*$.

5.2.1 String parametrisation of crystal bases

Let $\underline{w_0} = (\alpha_{i_1}, \dots, \alpha_{i_N})$ be the same reduced word for $w_0 \in W$ as the one we used to construct Bott-Samelson varieties. Then, given a vector b in the crystal basis $\mathcal{B}_\lambda$, we can associate to it a sequence of numbers that correspond to the maximal number of steps we can move up in the crystal graph from b , along the directions specified by the sequence $\underline{w_0}$. This procedure is called **string parametrisation**, first introduced by Littelmann in [Lit98]. The sequence of numbers $(a_1, \dots, a_N)$ thus obtained is called the **string parameters** of b .

Definition 5.2.1. Formally, we define the **string parameters** $(a_1, \dots, a_N)$ of b by

$$\begin{aligned} a_1 &= \max\{a \mid \tilde{E}_{\alpha_{i_1}}^a(b) \neq 0\}, \\ a_2 &= \max\{a \mid \tilde{E}_{\alpha_{i_2}}^a \tilde{F}_{\alpha_{i_1}}^{a_1}(b) \neq 0\}, \\ a_3 &= \max\{a \mid \tilde{E}_{\alpha_{i_3}}^a \tilde{E}_{\alpha_{i_2}}^{a_2} \tilde{F}_{\alpha_{i_1}}^{a_1}(b) \neq 0\}, \\ &\dots \end{aligned}$$

Write $\iota_{\underline{w_0}}(b) = (a_1, \dots, a_N)$. Then $\mathcal{S}_\lambda = \iota(\mathcal{B}_\lambda)$ is a subset of $\mathbb{Z}^N$. Denote by Λ^+ the set of all dominant (integral) weights. In the following theorem, Littelmann showed that one can concoct polyhedral geometry out of crystal bases using string parametrisation. (The following is stated more closely to the reformulation in [Kav15].)

Theorem 5.2.2. [Lit98, Proposition 1.5]

1. $\iota_{\underline{w_0}}$ injects $\mathcal{B}_\lambda$ into $\mathbb{Z}^N$. In particular, $\dim(V_\lambda) = \#\mathcal{S}_\lambda$.

2. Consider

$$\mathcal{S}_{\underline{w_0}} = \bigcup_{\lambda \in \Lambda^+} \{(\lambda, a) \mid a \in \mathcal{S}_\lambda\} \subset \Lambda^+ \times \mathbb{Z}_{\geq 0}^N.$$

Then $\mathcal{S}_{\underline{w_0}}$ consists of the lattice points of $\Lambda \times \mathbb{Z}^N$ lying in a rational closed convex polyhedral cone $\mathcal{C}_{\underline{w_0}}$. In particular, $\mathcal{C}_{\underline{w_0}}$ intersects the subspace $\{0\} \times \mathbb{R}^N$ only at the origin.

The **string polytope** $\Delta_{\underline{w_0}}(\lambda) \subset \mathbb{R}^N$ is obtained by taking a slice of $\mathcal{S}_{\underline{w_0}}$ at the level λ :

$$\Delta_{\underline{w_0}}(\lambda) = \{a \mid (\lambda, a) \in \mathcal{C}_{\underline{w_0}}\}.$$

As $\mathcal{C}_{\underline{w_0}}$ is a rational convex polyhedral cone and intersects $\{0\} \times \mathbb{R}^N$ only at the origin, $\Delta_{\underline{w_0}}(\lambda)$ is a convex polytope with rational vertices.

Therefore, the string parametrisation arranges $\mathcal{B}_\lambda$ into the integral points inside the string polytope $\Delta_{\underline{w_0}}(\lambda)$, and the number of such points counts $\dim(V_\lambda)$.

Since we care more about the dual crystal basis $\mathcal{B}_\lambda^*$, we investigate a dual string parametrisation defined on $\mathcal{B}_\lambda^*$:

Definition 5.2.3 ((Dual) string parametrisation in $\mathcal{B}_\lambda^*$). Define the string parametrisation map $\iota_{\underline{w_0}}^*: \mathcal{B}_\lambda^* \rightarrow \mathbb{Z}_{\geq 0}^N$ by $\iota_{\underline{w_0}}^* = (a_1, \dots, a_N)$, where

$$\begin{aligned} a_1 &= \max\{a \mid \tilde{F}_{\alpha_{i_1}}^a(b^*) \neq 0\}, \\ a_2 &= \max\{a \mid \tilde{F}_{\alpha_{i_2}}^a \tilde{F}_{\alpha_{i_1}}^{a_1}(b^*) \neq 0\}, \\ a_3 &= \max\{a \mid \tilde{F}_{\alpha_{i_3}}^a \tilde{F}_{\alpha_{i_2}}^{a_2} \tilde{F}_{\alpha_{i_1}}^{a_1}(b^*) \neq 0\}, \\ &\dots \end{aligned}$$

In [Kav15], the author takes this dual procedure to be the definition of string parametrisation and works with $\mathcal{B}_\lambda^*$ throughout. We will prove that this agrees with Littelmann's original definition using $\tilde{E}_\alpha$'s given above: applying the usual string parametrisation to $b \in \mathcal{B}_\lambda$ and applying the dual string parametrisation to $b^* \in \mathcal{B}_\lambda^*$ result in the same parameter $(a_1, \dots, a_N)$. Along the way, we will prove a result that generalises the string parametrisation to arbitrary vectors in the representation, not just the basis vectors.

We start with a suggestive lemma.

Lemma 5.2.4. *Let V be a finite-dimensional G -module and $\mathcal{B}$ be a basis consisting of weight vectors (not necessarily a crystal basis), and $\mathcal{B}^*$ the corresponding dual basis. Let $v \in V$ be a weight vector. Then $E_\alpha \cdot v = 0$ if and only if $F_\alpha \cdot v^* = 0$*

Proof. Suppose $E_\alpha \cdot v = 0$. It suffices to show that $(F_\alpha \cdot v^*)(w) = -v^*(F_\alpha \cdot w) = 0$ for $w \in \mathcal{B}$.

Now recall that within $\mathfrak{g} = \mathcal{L}(G)$, there is a copy of $\mathfrak{sl}_2$ corresponding to the root α : $\mathfrak{m}_\alpha := \text{Span}\{E_\alpha, F_\alpha, [E_\alpha F_\alpha]\} \cong \mathfrak{sl}_2$.

Restrict V to an $\mathfrak{m}_\alpha$ -representation. Then V decomposes into a direct sum of $\mathfrak{sl}_2$ -modules of highest weights $\mu_1, \dots, \mu_r$:

$$V = V(\mu_1) \oplus V(\mu_2) \oplus \dots \oplus V(\mu_r),$$

and $\mathcal{B}$ partitions into bases for the modules $V(\mu_i)$.

Then v is the highest weight vector for some $V(\mu_{i_0})$, and it is clear that $F_\alpha \cdot w$ can never have a v -component for $w \in \mathcal{B}$. This proves one direction, and the other direction is similar. $\square$

Next we show that in the definition of ϵ_α (respectively φ_α), we could have also used $\tilde{E}_\alpha$ instead of E_α (respectively $\tilde{F}_\alpha$ instead of F_α):

For $b \in \mathcal{B}_V$ and a simple root α , define $\tilde{\epsilon}_\alpha(b) := \max\{a \mid \tilde{E}_\alpha^a(b) \neq 0\}$. Similarly, define $\tilde{\varphi}_\alpha(b) := \max\{a \mid \tilde{F}_\alpha^a(b) \neq 0\}$.

Proposition 5.2.5. *Let V be a finite-dimensional G -module, $b \in \mathcal{B}_V$, and let α be a simple root. Then*

$$\epsilon_\alpha(b) = \tilde{\epsilon}_\alpha(b), \varphi_\alpha(b) = \tilde{\varphi}_\alpha(b).$$

Proof. We prove the statement for $\epsilon_\alpha(b)$. For $\varphi_\alpha(b)$, it is similar.

First notice that by the definition of $\tilde{E}_\alpha$, we have $E_\alpha \cdot c = 0$ if and only if $\tilde{E}_\alpha(b) \neq 0$ for any $c \in \mathcal{B}_V$.

Now proceed by induction on $\epsilon = \epsilon_\alpha(b)$.

By the construction of $\tilde{E}_\alpha$ we have

$$E_\alpha \cdot b = c_0 \tilde{E}_\alpha(b) + \sum_i c_i b_i$$

with $b_i \in \mathcal{B}_V$, $c_i \neq 0$, and $\epsilon_\alpha(\tilde{E}_\alpha(b)) = \epsilon - 1$, $\epsilon_\alpha(b_i) < \epsilon - 1$.

So by the induction hypothesis applied to $\tilde{E}_\alpha(b)$ we have

$$\epsilon_\alpha(\tilde{E}_\alpha \cdot b) = \tilde{\epsilon}_\alpha(\tilde{E}_\alpha \cdot b)$$

But $\epsilon_\alpha(\tilde{E}_\alpha \cdot b) = \epsilon - 1 = \epsilon_\alpha(E_\alpha \cdot b)$

So $\epsilon_\alpha(E_\alpha \cdot b) = \tilde{\epsilon}_\alpha(\tilde{E}_\alpha \cdot b)$, from which it follows that $\epsilon_\alpha(b) = \tilde{\epsilon}_\alpha(b)$. $\square$

Combining the previous two results (Lemma 5.2.4 and Proposition 5.2.5), we obtain

Corollary 5.2.6. *The two procedures in Definitions 5.2.1 and 5.2.3 give the same string parameters $(a_1, \dots, a_n)$.*

From this point onwards, we will work with $\mathcal{B}_\lambda^*$.

Using Proposition 5.2.5, we can extend $\iota_{\underline{w}_0}$ to the entire $V_\lambda^* \setminus \{0\}$:

Definition 5.2.7 (Generalised string parametrisation for V_λ^*). For $\sigma \in V_\lambda^* \setminus \{0\}$, define $\iota_{\underline{w}_0} = (a_1, \dots, a_N)$ by

$$\begin{aligned} a_1 &= \max\{a \mid F_{\alpha_{i_1}}^a \cdot \sigma \neq 0\}, \\ a_2 &= \max\{a \mid F_{\alpha_{i_2}}^a F_{\alpha_{i_1}}^{a_1} \cdot \sigma \neq 0\}, \\ a_3 &= \max\{a \mid F_{\alpha_{i_3}}^a F_{\alpha_{i_2}}^{a_2} F_{\alpha_{i_1}}^{a_1} \cdot \sigma \neq 0\}, \\ &\dots \end{aligned}$$

5.3 Main theorem: highest exponents equal string parameters

We are now ready to state the main result of this chapter.

Theorem 5.3.1. *For any $\sigma \in \Gamma(G/B, L_\lambda) \setminus \{0\}$,*

$$\iota_{\underline{w}_0}(\sigma) = v_{\underline{w}_0}(\sigma).$$

5.4 More on Schubert and Bott-Samelson varieties

We now need to look more closely at Schubert varieties and Bott-Samelson varieties to understand how the action of G is transferred when we go through $\tilde{X}_{\underline{w}_0} \rightarrow X_{\underline{w}_0} = G/B$ and $\tilde{\Phi}_{\underline{w}_0} : \mathbb{C}^N \rightarrow \tilde{X}_{\underline{w}_0}$ (as defined in section 4.4).

Recall that we have written $\underline{w}_0 = (\alpha_{i_1}, \dots, \alpha_{i_N})$. Define also a sequence of shorter words,

$$\underline{w}_k = (\alpha_{i_{k+1}}, \dots, \alpha_{i_N}),$$

for $0 \leq k \leq N$.

By the combinatorial description of the Bruhat order (Theorem 4.4.6), we have a filtration of Schubert varieties

$$X_N \subset X_{N-1} \subset \cdots \subset X_0 = X_{w_0}.$$

Correspondingly, we can also construct a sequence of Bott-Samelson varieties $\tilde{X}_N, \tilde{X}_{N-1}, \dots, \tilde{X}_0$, where $\tilde{X}_k = \tilde{X}_{\underline{w}_k}$, using the recipe in section 4.4.4. In particular, $\tilde{X}_0 = \tilde{X}_{\underline{w}_0}$. More explicitly, we have

$$\tilde{X}_k = P_{\alpha_{i_{k+1}}} \times \cdots \times P_{\alpha_{i_N}} / B^{N-k}.$$

We can identify $\tilde{X}_k$ as the subvariety of $\tilde{X}_0 = \tilde{X}_{\underline{w}_0}$ consisting of points $(1, 1, \dots, 1, p_{k+1}, p_{k+2}, \dots, p_N) \pmod{B^N}$ (i.e. the points $(p_1, \dots, p_N) \pmod{B^N}$ where $p_j = 1$ for $1 \leq j \leq k$). Through this identification, we fit our Bott-Samelson varieties also into a filtration

$$\tilde{X}_N \subset \tilde{X}_{N-1} \subset \cdots \subset \tilde{X}_0 = \tilde{X}_{\underline{w}_0}.$$

Then by Theorem 4.4.10, $\pi_{\underline{w}_0}$ maps $\tilde{X}_N$ birationally to X_N .

In a diagram, the setup is

$$\begin{array}{ccccccc} \tilde{X}_N & \subset & \tilde{X}_{N-1} & \subset & \cdots & \subset & \tilde{X}_0 = \tilde{X}_{\underline{w}_0} \\ & & & & & & \downarrow \pi_{\underline{w}_0} \\ X_N & \subset & X_{N-1} & \subset & \cdots & \subset & X_0 = G/B \end{array}$$

The Schubert varieties $X_1, \dots, X_n$ are not stable under the whole of G in general. But one can verify that for $0 \leq k \leq N-1$, X_k is invariant under (representatives of) $s_{\alpha_{i_{k+1}}}$, and thus under the opposite root subgroup $U_{\alpha_{i_{k+1}}}^-$. In particular, as in Section 3, we have an induced action by $\mathcal{L}(U_{\alpha_{i_{k+1}}}^-)$ on $\mathbb{C}(X_k)$, hence an action by $F_{\alpha_{i_{k+1}}} \in \mathcal{L}(U_{\alpha_{i_{k+1}}}^-)$.

In the same spirit, We can define an action of $P_{\alpha_{i_{k+1}}}$ on $\tilde{X}_k$ by multiplication on the left, i.e.:

$$p'_1 \cdot (p_1, \dots, p_d) = (p'_1 p_1, \dots, p_d).$$

This is well-defined on $\tilde{X}_k$ because if

$$(p_{k+1}, \dots, p_N) \sim (p_{k+1} b_{k+1}, b_{k+1}^{-1} p_{k+2} b_{k+2}, \dots, b_{N-1}^{-1} p_N b_N),$$

then

$$\begin{aligned} p'_{k+1} \cdot (p_{k+1}, \dots, p_N) &= (p'_{k+1} p_{k+1}, \dots, p_N) \\ &\sim (p'_{k+1} p_{k+1} b_{k+1}, b_{k+1}^{-1} p_{k+2} b_{k+2}, \dots, b_{N-1}^{-1} p_N b_N) \\ &= p'_{k+1} \cdot (p_{k+1} b_{k+1}, b_{k+1}^{-1} p_{k+2} b_{k+2}, \dots, b_{N-1}^{-1} p_N b_N). \end{aligned}$$

Remark. We point out immediately that there is no similar action of $P_{\alpha_{k+2}}, P_{\alpha_{k+3}}, \dots, P_{\alpha_N}$ on $\tilde{X}_k$ because the above calculation will not go through, and the action will not be well-defined. Consequently, there is no corresponding action by $F_{\alpha_{k+2}}, F_{\alpha_{k+3}}, \dots, P_{\alpha_N}$ on $\mathbb{C}(\tilde{X}_k)$ as in the next paragraph. The significance of this is explained after the next theorem. ∇

This restricts to an action of $U_{\alpha_{i_{k+1}}}^-$ on $\tilde{X}_k$ because for any α the minimal parabolic P_α contains s_α and U_α , and hence contains the opposite root subgroup U_α^- . In particular, we have an induced action by $\mathcal{L}(U_{\alpha_{i_{k+1}}}^-)$ on $\mathbb{C}(\tilde{X}_k)$, hence an action by $F_{\alpha_{i_{k+1}}} \in \mathcal{L}(U_{\alpha_{i_{k+1}}}^-)$.

One can check that the map $\pi_{w_0}: \tilde{X}_k \rightarrow X_k$ is equivariant with respect to the $U_{\alpha_{i_{k+1}}}^-$ -actions. This is important as we later need to compare the $F_{\alpha_{i_{k+1}}}$ -actions on $\mathbb{C}(\tilde{X}_k)$ and $\mathbb{C}(X_k)$.

The reasons for considering such filtrations become clear after the next theorem, which our proof depends crucially on.

Recall the map $\tilde{\Phi}_{w_0}: \mathbb{C}^N \rightarrow U_{\alpha_{i_1}}^- \times \dots \times U_{\alpha_{i_N}}^- \rightarrow X_{w_0}$ given by:

$$\tilde{\Phi}_{w_0}: (t_1, \dots, t_N) \mapsto (\exp(t_1 F_{\alpha_{i_1}}), \dots, \exp(t_N F_{\alpha_{i_N}})) \pmod{B^N}.$$

With this setup, we strengthen Proposition 4.4.11 to take into account the filtrations of varieties and $\mathfrak{g}$ -actions on them.

Theorem 5.4.1. 1. $\tilde{\Phi}_{w_0}$ is an open embedding whose image is a neighborhood $\tilde{\mathcal{U}}$ of the identity in $\tilde{X}_{w_0}$. In particular, it meets every $\tilde{X}_k$.

2. With local coordinates $t_1, \dots, t_N$ on $\tilde{\mathcal{U}}$ given by $\tilde{\Phi}_{w_0}$, the subvariety $\tilde{\mathcal{U}} \cap \tilde{X}_k$ is given by $t_1 = \dots = t_k = 0$.

3. The action of $F_{\alpha_{i_{k+1}}}$ on $\mathbb{C}(\tilde{X}_k)$ is given by $-\partial/\partial t_{k+1}$.

Proof. The first part of (1) is Proposition 4.4.11. The second part is because each $\tilde{X}_k$ contains $\tilde{X}_N$ which is the identity.

For (2), by definition $\tilde{X}_k$ is given by setting the first k coordinates to 1, so on $\tilde{X}_k \cap \tilde{\mathcal{U}}$ this is equivalent to setting $t_1 = \dots = t_k = 0$ by the definition of $\tilde{\Phi}_{w_0}$.

For (3), consider the action of $U_{\alpha_{i_{k+1}}}^-$ on $\tilde{\mathcal{U}} \cap \tilde{X}_k$. This is by definition

$$\begin{aligned} & \exp(s F_{\alpha_{i_{k+1}}}) \cdot (e, e, \dots, e, \exp(t_{k+1} F_{\alpha_{i_{k+1}}}), \dots, \exp(t_N F_{\alpha_{i_N}})) \\ &= (e, e, \dots, e, \exp((t_{k+1} + s) F_{\alpha_{i_{k+1}}}), \dots, \exp(t_N F_{\alpha_{i_N}})). \end{aligned}$$

In local coordinates $t_{k+1}, \dots, t_N$,

$$\exp(s F_{\alpha_{i_{k+1}}}) \cdot (t_{k+1}, \dots, t_N) = (s + t_{k+1}, \dots, t_N).$$

So the situation is very concrete, isomorphic to $\mathbf{G}_a$ acting on $\mathbb{C}^R$, where $R = N - k$, by

$$t \cdot (x_1, \dots, x_R) = (t + x_1, \dots, x_R).$$

We give details here to illustrate how such calculations are done. Recall that $\partial/\partial t$ is a basis for $\mathcal{L}(\mathbf{G}_a)$, and

$$\exp(s \frac{\partial}{\partial t}) = s.$$

So $\exp(s \frac{\partial}{\partial t})$ acts like $\exp(s F_{\alpha_{i_{k+1}}})$ from the above. So computing the action of $F_{\alpha_{i_{k+1}}}$ above is equivalent to computing the action of $\partial/\partial t$ here.

Let $f(x) = f(x_1, \dots, x_R)$ be a function on $\mathbb{C}^R$. By definition,

$$\begin{aligned} ((\partial/\partial t) \cdot f)(x) &= \frac{\partial}{\partial s} (\exp(s \frac{\partial}{\partial t}) \cdot f)(x)|_{s=0} \\ &= \frac{\partial}{\partial s} f(\exp(s \frac{\partial}{\partial t})^{-1} \cdot x)|_{s=0} \\ &= \frac{\partial}{\partial s} f(s^{-1} \cdot x)|_{s=0} \\ &= \frac{\partial}{\partial s} f(x_1 - s, x_2, \dots, x_R)|_{s=0} \\ &= -\frac{\partial f}{\partial x_1}(x). \end{aligned}$$

Hence the action by $\partial/\partial t$ is $-\partial/\partial x_1$, and the corresponding result for the action of $F_{\alpha_{i_{k+1}}}$ follows. $\square$

Let us emphasise the interpretation and limitations of the above theorem, whose second and third parts are indispensable in our later arguments. Consider for example the largest Bott-Samelson variety $\tilde{X}_0 = \tilde{X}_{w_0}$. The theorem tells us that the action of $F_{\alpha_{i_1}}$ on $\mathbb{C}(\tilde{X}_0 \cap \tilde{\mathcal{W}})$ is given by $-\partial/\partial t_1$. This is good news because the action of $F_{\alpha_{i_1}}$, which was differentiation on $\mathbb{C}(G/B)$, gets carried over to still a differentiation which is explicitly given.

However, as observed above, there is no action of $F_{\alpha_{i_2}}$ (or of $F_{\alpha_{i_3}}, \dots, F_{\alpha_{i_N}}$) on $\mathbb{C}(\tilde{X}_0 \cap \tilde{\mathcal{W}})$. To have an action of $F_{\alpha_{i_2}}$, we have to go down one step in the filtration, setting $t_1 = 0$, and then $F_{\alpha_{i_2}}$ corresponds to $-\partial/\partial t_2$, and so on.

We have now seen how the $\mathfrak{g}$ -action on a element of $\mathbb{C}(\tilde{X}_0 \cap \tilde{\mathcal{W}})$ is reflected by differentiating the rational function in $t_1, \dots, t_N$ representing it. Next, we begin linking this to the effects of F_α 's.

5.5 Computing string parameters along a filtration

Since we have to go down a filtration of Bott-Samelson varieties to interpret the successive applications of $F_{\alpha_{i_1}}, \dots, F_{\alpha_{i_N}}$ as applications of differential operators $-\partial/\partial t_1, \dots, -\partial/\partial t_N$, it is natural to ask whether we can also restrict attentions to subvarieties of $X_0 = G/B$ when computing the string parameters. It turns

out that this is true precisely in the way to make highest term valuations equal to string parametrisation (Theorem 5.3.1).

Let $\sigma \in \Gamma(G/B, L_\lambda) \setminus \{0\}$ and let its string parameters be $\iota_{\underline{w_0}}(\sigma) = (a_1, \dots, a_N)$. We show that when computing the string parameter a_k , we can restrict attention to the Schubert variety $X_{w_{k-1}}$.

Consider the map $\pi_w: \Gamma(G/B, L_\lambda) \rightarrow \Gamma(X_w, L_{\lambda|_{X_w}})$ which is the restriction of sections. Recall that B_λ^* is a basis of $\Gamma(G/B, L_\lambda)$ consisting of weight vectors. It is known (see e.g. [Kas93]) that there is a subset of B_λ^* , denoted $\mathcal{B}_\lambda(w)^*$, satisfying the following properties:

- The image of $\mathcal{B}_\lambda(w)^*$ under π_w is a basis for $\Gamma(X_w, L_{\lambda|_{X_w}})$. In particular, π_w is surjective, and $B_\lambda \setminus \mathcal{B}_\lambda(w)^*$ is a basis for $\ker(\pi_w)$.
- For any simple root α , $\mathcal{B}_\lambda(w)^* \cup 0$ is stable under $\tilde{F}_\alpha$.
- $\mathcal{B}_\lambda(w_0)^* = B_\lambda^*$.

We later make some remarks about how these are obtained via Demazure modules. For now, let us use this setup to give an alternative characterisation of $\iota_{\underline{w_0}}(\sigma)$.

We define an N -tuple of numbers $(e_1, \dots, e_N)$ by the following procedure.

Let

$$e_1 = a_1 = \max\{e \mid F_{\alpha_{i_1}}^e \cdot \sigma \neq 0\}.$$

Then setting $\sigma_1 = F_{\alpha_{i_1}}^{e_1} \cdot \sigma$, define

$$e_2 = \max\{e \mid \pi_{w_1}(F_{\alpha_{i_2}}^e \cdot \sigma_1) \neq 0\}.$$

Then setting $\sigma_2 = F_{\alpha_{i_2}}^{e_2} \cdot \sigma_1$, define

$$e_3 = \max\{e \mid \pi_{w_2}(F_{\alpha_{i_3}}^e \cdot \sigma_2) \neq 0\},$$

and so on.

We show that this procedure also gives the string parametrisation.

Theorem 5.5.1. *Let $\sigma \in \Gamma(G/B, L_\lambda) \setminus \{0\}$. Then $\iota_{\underline{w_0}}(\sigma) = (e_1, \dots, e_N)$.*

To prove this, we need a lemma, found as [Kas93, Lemma 3.3.3]:

Lemma 5.5.2. *Let s_α be a simple reflection and $w = s_\alpha w'$ with $\ell(w) = \ell(w') + 1$. Let $b^* \in B_\lambda^*$. If $\tilde{F}_\alpha(b^*) = 0$, then $b^* \in (\mathcal{B}_\lambda(w'))^*$.*

Remark. As in [Kav15, Lemma 3.13], we have stated this lemma in its dual version. Compared to [Kav15], we have also chosen to further simplify its statement to the suit our need. ∇

Proof of Theorem 5.5.1. By definition, already $a_1 = e_1$. Let $\sigma_1 = F_{\alpha_{i_1}}^{a_1} \cdot \sigma$. Let us show $a_2 = e_2$. Since if a section is non-zero over X_{w_1} , it certainly is non-zero over G/B , so $a_2 \geq e_2$. We need to show that conversely $\sigma_2 = F_{\alpha_{i_2}}^{a_2} \cdot \sigma_1$ is still not zero when restricted to X_{w_1} , i.e. $\sigma_2 \notin \ker(\pi_{w_1})$.

Now write $\sigma_1 = \sum_i c_i b_i^*$ with $c_i \neq 0$ and $b_i^* \in \mathcal{B}_\lambda^*$. Since $F_{\alpha_{i_1}} \cdot \sigma_1 = 0$, we have $\tilde{F}_{\alpha_{i_1}}(b_i^*) = F_{\alpha_{i_1}} \cdot b_i^* = 0$ for all i .

For any i , apply the above lemma with $w = w_0 = s_{\alpha_{i_1}} w_1$ to get that $b_i^* \in (\mathcal{B}_\lambda(w_1))^*$.

Since by the properties of $\mathcal{B}_\lambda(w)^* \cup 0$ it is stable under $F_{\alpha_{i_2}}$, as we apply $F_{\alpha_{i_2}}$ to get σ_2 , σ_2 stays in the span of $\mathcal{B}_\lambda(w_1)^*$. Since $\mathcal{B}_\lambda \setminus \mathcal{B}_\lambda(w)^*$ is a basis for $\ker(\pi_w)$, it follows that $\sigma_2 \notin \ker(\pi_{w_1})$, and $a_2 = e_2$ as desired.

Now σ_2 is in the span of $\mathcal{B}_\lambda(w_1)^*$, we can repeat the above argument, again applying the lemma, to prove $e_j = a_j$ for all j . $\square$

Remark. In fact, $\Gamma(X_w, L_{\lambda|X_w})$ can be identified with the dual $V_\lambda(w)^*$ of the Demazure module $V_\lambda(w)$, and the properties we listed came from results on the interplay between Demazure modules and crystal bases. Intuitively, the Demazure module $V_\lambda(w)$ is the subspace of V_λ consisting of weight spaces above the (1-dimensional, by W -invariance of dimensions) weight space of weight $w\lambda$. In particular, as $w_0\lambda$ is the lowest weight, $V_\lambda(w_0)$ is the whole of V_λ . ∇

5.6 Hurry now! The Proof!

We now tie up all loose ends and prove Theorem 5.3.1. Keep all notations of the preceding sections. The idea is simple: since actions by F_α correspond (partially) to differentiation of functions (Section 4.3) and then correspond further to (in our limited sense) differentiation of polynomials, the maximal number of times one can differentiate a section before reaching zero gives precisely its highest exponent as a polynomial.

Proof. Let $\sigma \in \Gamma(G/B, L_\lambda)$ and $f = f_\sigma = \sigma/\tau_\lambda$ as in Section 4.3. Let $\iota_{w_0}(\sigma) = (a_1, \dots, a_n)$ be its string parameters.

Theorem 5.5.1 tells us that we can compute $(a_1, \dots, a_n)$ along our filtration of Schubert varieties $X_N \subset X_{N-1} \subset \dots \subset X_0 = X_{w_0}$:

$$\begin{aligned} a_1 &= \max\{a \mid F_{\alpha_{i_1}}^a \cdot \sigma \neq 0\}, \\ a_2 &= \max\{a \mid (F_{\alpha_{i_2}}^a F_{\alpha_{i_1}}^{a_1} \cdot \sigma)|_{X_1} \neq 0\}, \\ &\dots \end{aligned}$$

By Proposition 4.3.1, we have

$$F_\alpha \cdot \sigma = (F_\alpha \cdot f)\tau_\lambda.$$

Therefore, we can further rewrite the string parameters:

$$\begin{aligned} a_1 &= \max\{a \mid F_{\alpha_{i_1}}^a \cdot f \neq 0\}, \\ a_2 &= \max\{a \mid (F_{\alpha_{i_2}}^a F_{\alpha_{i_1}}^{a_1} \cdot f)|_{X_1} \neq 0\}, \\ &\dots \end{aligned}$$

Now, as argued in Section 4.5, we know that f is regular on the open opposite cell $\mathcal{U}^-$ and pulling back via $\pi_{\underline{w}_0}$ yields a regular function $\tilde{f}$ on $\mathcal{U}$, which we expressed as a polynomial in $t_1, \dots, t_N$.

Moreover, as noted in Section 5.4, $\pi_{\underline{w}_0}: \tilde{X}_k \rightarrow X_k$ is equivariant with respect to the $U_{\alpha_{i_{k+1}}}^-$ -actions, hence with respect to the induced $\mathcal{L}(U_{\alpha_{i_{k+1}}}^-)$ -action. As $F_{\alpha_{i_{k+1}}} \in U_{\alpha_{i_{k+1}}}^-$, we have

$$\begin{aligned} a_1 &= \max\{a \mid F_{\alpha_{i_1}}^a \cdot \tilde{f} \neq 0\}, \\ a_2 &= \max\{a \mid (F_{\alpha_{i_2}}^a F_{\alpha_{i_1}}^{a_1} \cdot \tilde{f})|_{\tilde{X}_1} \neq 0\}, \\ &\dots \end{aligned}$$

Now we can use Theorem 5.4.1, which tells us that the action of $F_{\alpha_{i_{k+1}}}$ on $\tilde{X}_k$ is no other than $-\partial/\partial t_{k+1}$. Thus,

$$a_1 = \max\{a \mid (\partial/\partial t_1)^a \tilde{f} \neq 0\}.$$

Now $F_{\alpha_{i_1}}^{a_1} \cdot \tilde{f} = \pm(\partial/\partial t_1)^{a_1} \tilde{f}$. But signs clearly do not matter here. Writing $\tilde{f}_1 = (\partial/\partial t_1)^{a_1} \tilde{f}$, and we continue to get

$$\begin{aligned} a_2 &= \max\{a \mid (\partial/\partial t_2)^a \tilde{f}_1 \neq 0\}, \\ a_3 &= \max\{a \mid (\partial/\partial t_3)^a \tilde{f}_2 \neq 0\}, \\ &\dots \end{aligned}$$

Here we inductively defined $\tilde{f}_k = (\partial/\partial t_k)^{a_k} \tilde{f}_{k-1}$. Notice that at each k , $\tilde{f}_k$ already contains no t_k -term, since we have taken the highest t_k -derivative possible.

Then it is easy to see that a_k records precisely the highest t_k -exponent in $\tilde{f}_{k-1}$. So $(a_1, \dots, a_n)$ is precisely the highest term valuation of $\tilde{f}$ in the lexicographic order $t_1 > t_2 > \dots > t_N$. But this was how we defined $v_{\underline{w}_0}(\sigma)$, concluding the proof. $\square$

Bibliography

- [BK05] Michel Brion and S. Kumar. *Frobenius splitting methods in geometry and representation theory*. en. Progress in mathematics v. 231. OCLC: ocm54677804. Boston: Birkhäuser, 2005. ISBN: 978-0-8176-4191-7.
- [BK07] Arkady Berenstein and David Kazhdan. *Geometric and unipotent crystals II: From unipotent bicrystals to crystal bases*. arXiv:math/0601391. Mar. 2007. URL: <http://arxiv.org/abs/math/0601391> (visited on 02/27/2024).
- [Bor91] Armand Borel. *Linear Algebraic Groups*. en. Vol. 126. Graduate Texts in Mathematics. New York, NY: Springer New York, 1991. ISBN: 978-1-4612-6954-0 978-1-4612-0941-6. DOI: [10.1007/978-1-4612-0941-6](https://doi.org/10.1007/978-1-4612-0941-6). URL: <http://link.springer.com/10.1007/978-1-4612-0941-6> (visited on 04/20/2024).
- [Bri10] Michel Brion. “Introduction to actions of algebraic groups”. en. In: *Les cours du CIRM* 1.1 (2010), pp. 1–22. ISSN: 2108-7164. DOI: [10.5802/ccirm.1](https://ccirm.centre-mersenne.org/item/CCIRM_2010__1_1_1_0/). URL: https://ccirm.centre-mersenne.org/item/CCIRM_2010__1_1_1_0/ (visited on 04/11/2024).
- [FH04] William Fulton and Joe Harris. *Representation Theory*. en. Vol. 129. Graduate Texts in Mathematics. New York, NY: Springer New York, 2004. ISBN: 978-3-540-00539-1 978-1-4612-0979-9. DOI: [10.1007/978-1-4612-0979-9](https://doi.org/10.1007/978-1-4612-0979-9). URL: <http://link.springer.com/10.1007/978-1-4612-0979-9> (visited on 04/29/2024).
- [Ful96] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. en. 1st ed. Cambridge University Press, Dec. 1996. ISBN: 978-0-521-56724-4 978-0-521-56144-0 978-0-511-62624-1. DOI: [10.1017/CB09780511626241](https://doi.org/10.1017/CB09780511626241). URL: <https://www.cambridge.org/core/product/identifier/9780511626241/type/book> (visited on 02/20/2024).
- [Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. en. Vol. 9. Graduate Texts in Mathematics. New York, NY: Springer New York, 1972. ISBN: 978-0-387-90052-0 978-1-4612-6398-2. DOI: [10.1007/978-1-4612-6398-2](https://doi.org/10.1007/978-1-4612-6398-2). URL: [http:](http://)

- [//link.springer.com/10.1007/978-1-4612-6398-2](http://link.springer.com/10.1007/978-1-4612-6398-2) (visited on 04/29/2024).
- [Hum75] James E. Humphreys. *Linear Algebraic Groups*. Ed. by P. R. Halmos and C. C. Moore. Vol. 21. Graduate Texts in Mathematics. New York, NY: Springer, 1975. ISBN: 978-1-4684-9445-7 978-1-4684-9443-3. DOI: [10.1007/978-1-4684-9443-3](https://doi.org/10.1007/978-1-4684-9443-3). URL: <http://link.springer.com/10.1007/978-1-4684-9443-3> (visited on 03/06/2024).
- [Jan03] Jens Carsten Jantzen. *Representations of Algebraic Groups*. en. Google-Books-ID: Liqas0afjR0C. American Mathematical Soc., Jan. 2003. ISBN: 978-0-8218-3527-2.
- [Kah+14] Byeong Hoon Kahng et al. *Dual Perfect Bases and dual perfect graphs*. arXiv:1405.1820 [math]. May 2014. DOI: [10.48550/arXiv.1405.1820](https://doi.org/10.48550/arXiv.1405.1820). URL: <http://arxiv.org/abs/1405.1820> (visited on 02/27/2024).
- [Kas93] Masaki Kashiwara. “The crystal base and Littelmann’s refined Demazure character formula”. In: *Duke Mathematical Journal* 71.3 (Sept. 1993). Publisher: Duke University Press, pp. 839–858. ISSN: 0012-7094, 1547-7398. DOI: [10.1215/S0012-7094-93-07131-1](https://doi.org/10.1215/S0012-7094-93-07131-1). URL: <https://projecteuclid.org/journals/duke-mathematical-journal/volume-71/issue-3/The-crystal-base-and-Littelmanns-refined-Demazure-character-formula/10.1215/S0012-7094-93-07131-1.full> (visited on 04/29/2024).
- [Kav15] Kiumars Kaveh. “Crystal bases and Newton–Okounkov bodies”. In: *Duke Mathematical Journal* 164.13 (Oct. 2015). Publisher: Duke University Press, pp. 2461–2506. ISSN: 0012-7094, 1547-7398. DOI: [10.1215/00127094-3146389](https://doi.org/10.1215/00127094-3146389). URL: <https://projecteuclid.org/journals/duke-mathematical-journal/volume-164/issue-13/Crystal-bases-and-NewtonOkounkov-bodies/10.1215/00127094-3146389.full> (visited on 04/29/2024).
- [KK12] Kiumars Kaveh and A. G. Khovanskii. *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*. arXiv:0904.3350 [math]. Mar. 2012. DOI: [10.48550/arXiv.0904.3350](https://doi.org/10.48550/arXiv.0904.3350). URL: <http://arxiv.org/abs/0904.3350> (visited on 03/06/2024).
- [Lit98] P. Littelmann. “Cones, crystals, and patterns”. en. In: *Transformation Groups* 3.2 (June 1998), pp. 145–179. ISSN: 1083-4362, 1531-586X. DOI: [10.1007/BF01236431](https://doi.org/10.1007/BF01236431). URL: <http://link.springer.com/10.1007/BF01236431> (visited on 02/20/2024).
- [LM09] Robert Lazarsfeld and Mircea Mustață. “Convex bodies associated to linear series”. en. In: *Annales scientifiques de l’École Normale Supérieure* 42.5 (2009), pp. 783–835. ISSN: 1873-2151. DOI: [10.24033/asens.2109](https://doi.org/10.24033/asens.2109). URL: http://www.numdam.org/item/ASENS_2009_4_42_5_783_0/ (visited on 05/02/2024).

- [Lur] Jacob Lurie. “A PROOF OF THE BOREL-WEIL-BOTT THEOREM”. en. In: ().
- [Ma] Man Shun John Ma. “A PROOF OF BOREL-WEIL-BOTT THEOREM”. en. In: ().
- [Mag98] P. Magyar. “Schubert polynomials and Bott-Samelson varieties”. en. In: *Commentarii Mathematici Helvetici* 73.4 (Nov. 1998), pp. 603–636. ISSN: 0010-2571, 1420-8946. DOI: [10.1007/s000140050071](https://doi.org/10.1007/s000140050071). URL: <http://www.emis-ph.org/doi/10.1007/s000140050071> (visited on 03/27/2024).
- [Oko00] Andrei Okounkov. *Why would multiplicities be log-concave ?* arXiv:math/0002085. Feb. 2000. DOI: [10.48550/arXiv.math/0002085](https://arxiv.org/abs/math/0002085). URL: <http://arxiv.org/abs/math/0002085> (visited on 05/02/2024).
- [Spi99] Michael Spivak. *A comprehensive introduction to differential geometry. vol. 2 / Michael Spivak*. eng. 3rd ed. Houston, Tex.: Publish or Perish, Inc., 1999. ISBN: 978-0-914098-71-3.
- [Spr98] T. A. Springer. *Linear Algebraic Groups*. en. Boston, MA: Birkhäuser Boston, 1998. ISBN: 978-0-8176-4839-8 978-0-8176-4840-4. DOI: [10.1007/978-0-8176-4840-4](https://link.springer.com/10.1007/978-0-8176-4840-4). URL: <http://link.springer.com/10.1007/978-0-8176-4840-4> (visited on 04/07/2024).
- [WY23] Alexander Woo and Alexander Yong. *Schubert geometry and combinatorics*. arXiv:2303.01436 [math]. Mar. 2023. DOI: [10.48550/arXiv.2303.01436](https://arxiv.org/abs/2303.01436). URL: <http://arxiv.org/abs/2303.01436> (visited on 04/29/2024).