

Heights on Toric Varieties

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Chapter 1

Introduction

The theory of heights is an essential and powerful tool in Diophantine geometry. Arakelov geometry provides a framework where heights can be defined and studied on arbitrary varieties. Following [BPS14], the goal of this expository paper is to develop a machinery that makes explicit, in combinatorial and convex-geometric terms, the calculations of Arakelov heights on toric varieties.

We now give a brief survey of the history and motivations of heights and Arakelov geometry, partly following [Jon04].

Intuitively, the height of a point on a variety measures the “complexity” of this point. For example, on $\mathbb{P}^n(\mathbb{Q})$, we have the naïve height $h: \mathbb{P}^n(\mathbb{Q}) \rightarrow \mathbb{R}$: if we write $p = [a_0 : \cdots : a_n] \in \mathbb{P}^n(\mathbb{Q})$ such that $a_i \in \mathbb{Z}$ and $\gcd(a_1, \dots, a_n) = 1$, then define its height by

$$h(p) = \max\{|a_0|, \dots, |a_n|\}.$$

Note that this has the important property, called the *Northcott property* and shared by many later variants of heights, that given any constant C , the set of points p with $h(p) < C$ is finite.

An early application of heights was the proof of Mordell’s Theorem for elliptic curves in 1922, later generalised by the Mordell-Weil Theorem to abelian varieties over number fields (see, for example, [Sil86]).

Theorem 1.1 (Mordell’s Theorem, 1922). Let E be an elliptic curve over $\mathbb{Q}$. Then its rational points $E(\mathbb{Q})$ form a finitely generated abelian group.

Around the same time, Mordell posed the following conjecture for the higher genus cases.

Conjecture 1.2 (The Mordell Conjecture, 1922; Faltings’ Theorem, 1983). Let X be a smooth projective curve of genus $g \geq 2$ over a number field K . Then $X(K)$ is a finite set.

An analogous statement for smooth projective curves over function fields of characteristic 0 was proved in the 1960s by Manin and Grauert. Later, Parshin

and Arakelov came up with a different proof to the function field case, whose essential ingredient is a height function on the set of rational points with a Northcott property.

The nature of Parshin and Arakelov's height function is *intersection-theoretic*. Given a function field K and a curve X/K , an essential step of their argument is to choose a model

$$p: \mathcal{X} \rightarrow B,$$

where $\mathcal{X}$ is a proper algebraic surface, B is a smooth projective curve with $K(B) = K$, and the generic fibre $\mathcal{X}_{K(B)}$ is isomorphic to X . Now, rational points on X will correspond to sections s of the fibration p . Then, using classical intersection theory on $\mathcal{X}$, the height of a rational point is defined in terms of an intersection number involving the corresponding section. Using inequalities on canonical class of this surface, such intersection numbers can be bounded, thereby obtaining a bound on the heights of rational points and finishing the proof.

It is then very natural to try to generalise this strategy to the case of number fields. Let K be a number field and $\mathcal{O}_K$ be its ring of integers. For simplicity, we work with the case $K = \mathbb{Q}$, $\mathcal{O}_K = \mathbb{Z}$. For a curve $X/\mathbb{Q}$, one can again associate a model, which is a proper morphism

$$p: \mathcal{X} \rightarrow \operatorname{Spec} \mathcal{O}_K = \operatorname{Spec} \mathbb{Z},$$

where the fibres are curves, and the generic fibre is isomorphic to X . Then again, rational points correspond to sections.

But the issue now is that unlike in the previous case, $\mathcal{X}$ is not compact, and we cannot expect a well-behaved intersection theory on $\mathcal{X}$. Now, the closed points in $\operatorname{Spec} \mathbb{Z}$ correspond to primes in $\mathbb{Z}$, and hence to the p -adic (non-archimedean) absolute values. Ostrowski's theorem says that the absolute values on $\mathbb{Z}$ are precisely the usual (archimedean) absolute value $|\cdot|_\infty$ and the p -adic absolute values $|\cdot|_p$. To solve the compactification problem, Arakelov's ingenious idea is that we should formally adjoin the fibre over the archimedean place $|\cdot|_\infty$ ("the fibre at infinity") by considering the Riemann surface naturally associated with the complex points $X(\mathbb{C})$ of X ; then the pair $(\mathcal{X}, X(\mathbb{C}))$ should behave like a compact surface.

In this setup, one can define the notion of an *arithmetic divisor*, which consists of a Weil divisor on $\mathcal{X}$ together with some component "at infinity". Such arithmetic divisors form a group $\widehat{\operatorname{Div}}(\mathcal{X})$, and Arakelov developed a symmetric bilinear intersection pairing on $\widehat{\operatorname{Div}}(\mathcal{X})$ in his influential paper "An intersection theory for divisors on an arithmetic surface" ([Ara74]).

This theory did not directly allow a translation of the function field proof into the number field case, due to the lack of arithmetic analogues of the inequalities used in the function field setting, and the Mordell Conjecture was eventually proved by Faltings in 1983 using somewhat different ideas. However, it did become the starting point of Arakelov geometry, which has since evolved into a rich subject.

In higher dimensions, this philosophy was developed by Bost, Gillet, and Soulé into an arithmetic intersection theory and a general height theory [BGS94]. Given a smooth projective variety X defined over a number field K , the data required for this setup is still $(X, X(\mathbb{C}))$, where $p: X \rightarrow \text{Spec } \mathcal{O}_K$ is a proper model like before. The arithmetic intersection theory is then formulated in terms of *arithmetic line bundles*, or *metrised line bundles*: An arithmetic line bundle $\bar{L}$ is a pair $(\mathcal{L}, \|\cdot\|)$, where $\mathcal{L}$ is a line bundle on X restricting to a line bundle L on X , and $\|\cdot\|$ is a *smooth metric* on the complex line bundle associated to L on $X(\mathbb{C})$.

This approach has some technical shortcomings. For one, it restricts us to smooth varieties and smooth metrics. But many toric varieties are not smooth, and many natural metrics in toric settings are not smooth, either. Moreover, it requires the existence of an integral model, and it puts the archimedean and non-archimedean places on different footings.

In the 1990s, Zhang introduced the adelic point of view into height theory [Zha95], which removes the above restrictions and puts the archimedean and non-archimedean places on more equal footings. In particular, this setting considers uniform limits of semipositive metrics, which include the natural toric metrics. The adelic viewpoint is further developed in subsequent works such as [Gub02], [Gub03], [Cha06].

In this formulation of heights, the height function, called the *global height*, consists of local contributions from places, called *local heights*. This formalism is the one the authors of [BPS14] work in to develop the arithmetic geometry of toric varieties, and it is the one we will take in this exposition.

A brief introduction to this formalism is given in our Chapter 3. As we will see there, the intersection-theoretic nature of heights is less explicit in this setting, but the reader will surely catch glimpses of its shadows. Moreover, we will comment on how to translate the data of the Bost–Gillet–Soulé setup into this language, after which the two formalisms will produce the same outputs.

The local heights will be defined as certain integrals on the corresponding analytic spaces (complex or Berkovich analytic spaces, depending on the place) and are a priori not straightforward to compute. The merit of the theory developed by [BPS14] is that it makes this computation very explicit in the case of toric varieties.

By extending the standard dictionary between toric and convex geometries, the authors can express the (toric) local height of a toric variety X with respect to a metrised line bundle $\bar{L}$ as the integral of a concave function over the polytope Δ defining X (Theorem 5.5):

$$h_{\bar{L}}^{\text{tor}}(X) = (n+1)! \int_{\Delta} \psi^{\vee} d\text{vol}_M.$$

Here, ψ is a concave function determined by the metric $\|\cdot\|$ on L , and $\psi^{\vee}$ is its Legendre-Fenchel dual (see Chapter 2). For a more precise statement, see Theorem 5.5.

Combinatorial / convex data	Toric object
<i>Algebraic Geometry</i>	
fan Σ	toric variety X_Σ
polytope Δ	projective toric variety X_Δ
piecewise linear functions $\Psi: \Sigma \rightarrow \mathbb{R}$	T-Cartier divisors D_Ψ
<i>Arithmetic Geometry</i>	
continuous functions $\psi: N_{\mathbb{R}} \rightarrow \mathbb{R}$	toric metrics on $\mathcal{O}(D_\Psi)$
extending to N_Σ	
concave functions $\psi: N_{\mathbb{R}} \rightarrow \mathbb{R}$	semipositive toric metrics
with $ \psi - \Psi $ bounded	on $\mathcal{O}(D_\Psi)$
Monge–Ampère measure of ψ	Chambert–Lair measure

To keep our exposition grounded, throughout we will keep in mind the example of $\mathbb{P}_{\mathbb{Q}}^n$. This will occur as multiple examples throughout the entire text, as new concepts are introduced. Such examples shall be our *guiding examples*, and will be highlighted like this current paragraph. One way to read this exposition is to first locate all such guiding examples, and work backwards to learn the necessary machineries.

This exposition is organised as follows.

In Chapter 2, we give an introduction to the theory of Legendre–Fenchel duality of concave functions and their associated convex decompositions. It provides the convex geometry background required for this exposition, but it is also of independent interest, because this duality of convex decompositions generalises the polytope–normal–fan correspondence, familiar to anyone who has studied polyhedral or toric geometry.

In Chapter 3, we give a concise treatment of Arakelov heights, as discussed above. We aim to equip the reader with a working understanding of the key notions of this formalism, without necessarily giving the full details.

In Chapters 4 and 5, we start to specialise in the setting of toric varieties, using the background we have built up in the previous chapters. In Chapter 4, we develop the important Theorems 4.11 and 4.14 which transport the metric data to the Euclidean space $N_{\mathbb{R}}$. Finally, in Chapter 5, the material developed in previous sections culminates in the local toric height formula (Theorem 5.5), and we conclude with its application to the computation of global heights.

Chapter 2

The Legendre-Fenchel Duality

The purpose of this section is two-fold. On the one hand, the convex geometry is of intrinsic interest: Given a polytope P , the polytope-normal-fan correspondence states that its faces $F \subset P$ are in inclusion-reversing correspondence with the cones of its normal fan. This familiar fact from polyhedral geometry, used in toric geometry and many contexts, can be upgraded to a duality between *concave functions*, inducing a correspondence of *convex decompositions*, using the theory of Legendre-Fenchel Duality. We shall give a concise and accessible introduction to this interesting theory.

On the other hand, many of the notions we develop are important and necessary for studying heights on toric varieties.

2.1 Basic definitions

We first set up some notations to be used throughout this chapter.

Throughout the paper, we will use the following standard setup of convex geometry in the context of toric geometry.

Notation 2.1. Let $N \simeq \mathbb{Z}^n$ be a lattice and $M = \text{Hom}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$ be its dual lattice. Declare $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^n$, $M_{\mathbb{R}} = M \otimes \mathbb{R} \simeq \mathbb{R}^n$. Denote by $(\cdot, \cdot)$ both the natural perfect pairings $M \times N \rightarrow \mathbb{Z}$ and $M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$.

We assume the reader is familiar with the basic notions of polyhedral geometry, such as polytopes, cones, fans. These can be found in [Ful93, Chapter 1].

We briefly recall some as follows. For the sake of definitions, everything below is inside $N_{\mathbb{R}}$.

A *polyhedron* is a finite intersection of half-spaces. A *polytope* is a bounded polyhedron.

A *polyhedral cone*, or in short a *cone*, is a polyhedron σ such that $\lambda\sigma = \sigma$ for any $\lambda > 0$. Equivalently, it is the non-negative linear combination $\text{cone}(v_1, \dots, v_l)$ of some vectors $v_1, \dots, v_l \in N_{\mathbb{R}}$. A cone is *rational* if the generators v_i can be taken from the lattice N .

A polyhedron or cone is *strongly convex* if it contains no 1-dimensional subspaces of $N_{\mathbb{R}}$. A *fan* is a “compatible” collection Σ of strongly convex rational cones, defined more precisely below.

Next, let us recall some definitions of general convex geometry, following [BPS14, Chapter 2.1]. For the use of any convex geometry terms we have not defined, we refer the reader to [BPS14, Chapter 2.1].

Let $C \subset N_{\mathbb{R}}$ be a convex set.

Definition 2.2. The *affine hull* $\text{aff}(C)$ is the minimal affine space containing C . The *relative interior* of C is the topological interior of C with respect to $\text{aff}(C)$ and is denoted by $\text{relint}(C)$.

A subset $F \subset C$ is a *face* of C if whenever a line segment $\overline{u_1 u_2} \subset C$ such that $\text{relint}(\overline{u_1 u_2})$ meets F , we have $\overline{u_1 u_2} \subset F$. This differs from the usual definition of faces of polytope.

That definition leads to the definition of an exposed face of C : $F \subset C$ is an *exposed face* of C if there exists $m \in M_{\mathbb{R}}$ such that F minimises the pairing with m :

$$F = \{v \in C \mid (m, v) \leq (m, u) \quad \forall u \in C\}.$$

An exposed face is always a face. For polyhedrons, faces and exposed faces are the same.

A *convex subdivision* is a collection Π of convex sets such that

1. if $C \in \Pi$, then every face of C is also in Π .
2. The intersection of any two convex sets in Π is either empty or a face of both.

Π is called a *convex decomposition* if it only satisfies (2).

The *support* $|\Pi|$ of a convex decomposition Π is the union of its members, and Π is said to be *complete* if $|\Pi| = N_{\mathbb{R}}$.

A *polyhedral complex* is a finite convex subdivision whose members are polyhedra. A *fan* is a polyhedral complex whose members are strongly convex rational cones.

Next, we define some notions for concave functions.

Set $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ with the natural order and arithmetic operations.

Definition 2.3. A function $f: \mathbb{R}^n \rightarrow \underline{\mathbb{R}}$ is concave if

$$f(tu_1 + (1-t)u_2) \geq tf(u_1) + (1-t)f(u_2)$$

for all $u_1, u_2 \in \mathbb{R}^n$ and $t \in (0, 1)$, and f is not identically $-\infty$.

The *effective domain* $\text{dom}(f)$ of a concave function f is the subset of $\mathbb{R}^n$ on which f takes finite values. This is a convex set.

We say that a concave function f is closed if it is upper semicontinuous, i.e. $f^{-1}([a, \infty)) \subset \mathbb{R}^n$ is closed for all $a \in \mathbb{R}$. Given an arbitrary concave function f , there exists a unique minimal concave function over f , called its closure and denoted by $\text{cl}(f)$.

Let $C \subset \mathbb{N}_{\mathbb{R}}$ be a polyhedron. A function $f: C \rightarrow \mathbb{R}$ is said to be *piecewise affine* if C is a finite union of closed subsets C_i such that $f|_{C_i}$ is an affine function for each i . A concave function $f: \mathbb{N}_{\mathbb{R}} \rightarrow \underline{\mathbb{R}}$ is *piecewise affine* if $\text{dom}(f)$ is a polyhedron and $f|_{\text{dom}(f)}$ is piecewise affine.

2.2 The polytope-normal-fan correspondence

Our starting point is the familiar polytope-normal-fan correspondence, which we recall now.

Definition 2.4. Given a face $F \subset P$, we define its normal cone $\sigma_F = \{v \in \mathbb{N}_{\mathbb{R}} \mid (f, v) \leq (m, v) \text{ for all } f \in F, m \in P\}$, i.e. the set of functionals v attaining its minimum on P at F .

The (inner) normal fan Σ_P of a polytope P consists of the cones $\{\sigma_F\}_{F \subset P}$ as F ranges over all faces.

This indeed defines a fan. The assignment $F \mapsto \sigma_F$ is an inclusion-reversing bijection. The inverse is given by

$$\sigma_F \mapsto \bigcap_{v \in \sigma_F} \{m \in P \mid (m, v) = \inf_{m' \in P} (m', v)\} = \bigcap_{v \in \sigma_F} \text{argmin}_{m \in P} (m, v).$$

In other words, we simply take the points of P which evaluate to the minimum when paired with any vector in σ_F .

It is enough, in fact, to take any vector v_0 in the relative interior of σ_F , i.e.

$$\bigcap_{v \in \sigma_F} \text{argmin}_{m \in P} (m, v) = \text{argmin}_{m \in P} (m, v_0)$$

if $v_0 \in \text{relint}(\sigma_F)$.

We can give an alternative characterisation of normal cone σ_F .

Definition 2.5. Given a face $F \subset P$ of a polytope P , we define the *angle of P at F* to be

$$\angle(F, P) := \{t(m - f) \mid t \geq 0, m \in P, f \in F\} \subset M_{\mathbb{R}}.$$

Lemma 2.6. We have

$$\sigma_F = \angle(F, P)^{\vee}.$$

Proof. This follows directly from definitions. □

2.3 Legendre-Fenchel duality of concave functions

Notice that the inverse map defined above depends on the knowledge of P . There is no way to recover P from the fan Σ_P alone because Σ_P does not remember the positions of the supporting hyperplanes.

We can rectify this problem by remembering these support data, which we describe now.

Let $C \subset \mathbb{N}_{\mathbb{R}}$ be a closed convex set. Define the *indicator function* ι_C of C to be 0 on C and $-\infty$ otherwise. The *support function* of C is given by

$$\Psi_C(x) = \inf_{u \in C} (x, u).$$

Specialise to the case when C is a polytope $P = \text{conv}(m_1, \dots, m_k)$ with vertices m_i . Now ι_P tells us about P , and we claim that Ψ_P remembers both the fan Σ_P and our polytope P .

Indeed, note that for any vertex m_i , σ_{m_i} is a maximal cone in Σ_P , and Ψ_P is linear on σ_{m_i} , given by $u \mapsto (u, m_i)$. Hence the linearity domains of Ψ_P give us Σ_P .

Moreover, we can recover P as an intersection of half-spaces:

$$P = \bigcap_{v \in \mathbb{N}_{\mathbb{R}}} \{m \in M_{\mathbb{R}} \mid (v, m) \geq \Psi_P(v)\}.$$

In fact, it suffices to let v only range over the generators of 1-dimensional cones in Σ_P .

We can generalise this construction to *any* concave function beyond the indicator function by modifying the domain we work in.

Definition 2.7. Let f be a concave function. We define the *hypograph* of f to be the set

$$\text{hypo}(f) = \{(u, \lambda) \in \mathbb{N}_{\mathbb{R}} \times \mathbb{R} \mid \lambda \leq f(u)\}.$$

Since f is concave, $\text{hypo}(f)$ is a convex set. We can consider its support function $\Psi_{\text{hypo}(f)}: M_{\mathbb{R}} \times \mathbb{R} \rightarrow \mathbb{R}$. We restrict our attention to its values on vectors $(m, -1) \in M_{\mathbb{R}} \times \mathbb{R}$, corresponding to *non-vertical* hyperplanes.

Definition 2.8. Let f be a concave function on $\mathbb{N}_{\mathbb{R}}$. The *Legendre-Fenchel dual* $f^\vee$ is defined by

$$\begin{aligned} f^\vee(m) &= \Psi_{\text{hypo}(f)}(m, -1) \\ &= \inf_{u \in \mathbb{N}_{\mathbb{R}}} ((m, u) - f(u)). \end{aligned}$$

Notice that since $\text{hypo}(\iota_C) = C \times \mathbb{R}_{\leq 0}$ for a convex set C , we have $\iota_C^\vee$ is nothing but the support function Ψ_C .

Given the recovery of P from its support function above, one may expect to be able to recover f from $f^\vee$. Indeed, $f \mapsto f^\vee$ defines a duality of *closed* concave

functions: one has $f^{\vee\vee} = \text{cl}(f)$, so, in particular, $f^{\vee\vee} = f$ if f is closed ([Roc70, Corollary 12.2.1]).

We end with a notation.

Definition 2.9. The *stability set* $\text{stab}(f) := \text{dom}(f^{\vee})$ is defined to be the effective domain of $f^{\vee}$.

2.4 Legendre-Fenchel correspondence of convex decompositions

In the motivating example above, we saw that for a polytope P , its normal fan Σ_P can be recovered from the support function Ψ_P by considering its linearity domains. In general, it becomes very interesting to study the linearity domains of concave functions f and how this interacts with the Legendre-Fenchel duality. Here, by linearity domains, we simply mean subsets over which f is affine.

We begin by considering how to define the “gradient” of a (not necessarily smooth) concave function using a simple example.

Example 2.10. Consider $f(x) = \min\{x + 1, 2x - 3\}$. There is no unique number we can reasonably assign to be $f'(4)$, since f is not differentiable there. However, if we allow the derivative to be multi-valued, it is sensible to say that $f'(4)$ “can” be any value between 1 and 2, and we simply declare the interval $[1, 2]$ to be its derivative at $x = 4$. $\diamond$

The above example shows that concavity is a good global regularity condition to make multivalued derivatives a well-behaved notion.

Definition 2.11. Let f be a concave function on $N_{\mathbb{R}}$. Define the *sup-differential* of f at $u \in N_{\mathbb{R}}$ to be the set

$$(\partial f)(u) = \{x \in M_{\mathbb{R}} \mid (x, v - u) \geq f(v) - f(u)\}$$

if $u \in \text{dom}(f)$, and $\partial f(u) = \emptyset$ otherwise. We may sometimes regard ∂f as a multi-valued function in this way and use $\text{im}(\partial f)$ in the obvious way.

These are simply the defining functionals of the hyperplanes at u which dominate f everywhere. If $\partial f(u) \neq \emptyset$, we say that f is sup-differentiable at u , and denote the collection of all such points by $\text{dom}(\partial f)$. By [Roc70, Theorem 23.4], f is sup-differentiable almost everywhere:

$$\text{relint}(\text{dom}(f)) \subset \text{dom}(\partial f) \subset \text{dom}(f).$$

In the above example, we have $\partial f(4) = [1, 2]$, while $\partial f(x) = f'(x)$ agrees for $x \neq 4$. In general, we have $\partial f(u) = \nabla f(u)$ if f is differentiable at u .

The simple example above becomes somewhat surprising if we notice how ∂f interacts with $f^{\vee}$.

Example 2.12. It is easy to see that $\text{dom}(f^\vee) = [1, 2]$; more specifically,

$$f^\vee(\alpha) = \begin{cases} 4\alpha - 5, & 1 \leq \alpha \leq 2 \\ -\infty, & \text{otherwise} \end{cases}$$

We further compute its sup-differential for $\alpha \in \text{dom}(f^\vee)$.

$$\partial f^\vee(\alpha) = \begin{cases} [4, \infty), & \alpha = 1 \\ \{4\}, & 1 < \alpha < 2 \\ (-\infty, 4], & \alpha = 2 \end{cases}$$

The striking feature here is that $\partial f^\vee(\alpha)$ picks out the intervals over which f has slope α (i.e., the set of points u such that $\alpha \in \partial f(u)$), decomposing $\text{dom}(f)$ into the linearity domains, as α ranges over $\text{dom}(f^\vee)$.

Similarly, ∂f also gives a (trivial) decomposition of $\text{dom}(f^\vee)$. $\diamond$

This becomes slightly less miraculous if we examine more closely the relations between the sup-differential and the Legendre-Fenchel dual.

We begin by relating $\text{stab}(f) = \text{dom}(f^\vee)$ to the sup-differential ∂f .

Proposition 2.13. If there exists $u \in \text{dom}(f)$ such that $m \in \partial f(u)$, then $m \in \text{stab}(f)$. In other words, $\text{im}(\partial f) \subset \text{stab}(f)$.

Proof. If $m \in \partial f(u)$, then for all v , $f(v) - (m, v) \leq f(u) - (m, u)$, which means that the infimum in $f^\vee(m) = \inf_{v \in \mathbb{N}_\mathbb{R}} ((m, v) - f(v))$ is attained at u . In particular, $f^\vee(m) > -\infty$. $\square$

The converse is not true: for example, if $f(u) = -\exp(-u)$, then $0 \in \text{stab}(f)$, but $f'(u) \neq 0$ for all u . However, since $\text{im}(\partial f^\vee) = \text{dom}(\partial f)$ by Proposition 2.14 below, the almost-everywhere sup-differentiability of $f^\vee$ implies that

$$\text{relint}(\text{stab}(f)) \subset \text{im}(\partial f) \subset \text{stab}(f).$$

The proof above motivates us to consider how far $f(u) - (m, u)$ is from achieving $f^\vee(m) = \inf_{v \in \mathbb{N}_\mathbb{R}} ((m, v) - f(v))$. We define a pairing $P_f: M_\mathbb{R} \times \mathbb{N}_\mathbb{R} \rightarrow \mathbb{R}$ by

$$P_f(m, u) = \inf_{v \in \mathbb{N}_\mathbb{R}} ((m, v) - f(v)) - ((m, u) - f(u)) = f^\vee(m) + f(u) - (u, m).$$

This has the virtue of being symmetric in f and $f^\vee$, so it interacts well with duality. Note that $P_f(m, u) \leq 0$ by construction, and we have

Proposition 2.14. Let f be a closed concave function. Then

$$m \in \partial f(u) \iff P_f(m, u) = 0 \iff u \in \partial f^\vee(m).$$

Proof. The first equivalence follows directly from definitions, similar to the proof of Proposition 2.13. The second equivalence follows from the first equivalence and $f = f^{\vee\vee}$. $\square$

This implies in particular that $\text{im}(\partial f^\vee) = \text{dom}(\partial f)$. More importantly, since $\partial f^\vee(\mathbf{m}) = \{\mathbf{u} \in N_{\mathbb{R}} \mid f(\mathbf{u}) = (\mathbf{m}, \mathbf{u}) - f^\vee(\mathbf{m})\}$, we know that $f|_{\partial f^\vee(\mathbf{m})}$ is an affine function with linear part determined by $\mathbf{m}$.

Hence, we have a decomposition of $\text{dom}(\partial f)$ into the linearity domains of f :

$$\text{dom}(\partial f) = \bigcup_{\mathbf{m} \in M_{\mathbb{R}}} \partial f^\vee(\mathbf{m}).$$

Definition 2.15. Let f be a closed concave function on $N_{\mathbb{R}}$. For $\mathbf{m} \in M_{\mathbb{R}}$, write $C_{\mathbf{m}} = \partial f^\vee(\mathbf{m})$. We denote by $\Pi(f)$ the collection of all the sets $C_{\mathbf{m}}$.

We next prove that $\Pi(f)$ is a *convex decomposition*, as defined in Section 2.1. We will call it the *convex decomposition associated with f* .

For this, we recall that the Legendre-Fenchel dual $f^\vee$ is nothing but the support function of $\text{hypo}(f)$.

Proposition 2.16. Let f be a closed concave function on $N_{\mathbb{R}}$. The projection onto $N_{\mathbb{R}}$ induces a bijection between the non-vertical exposed faces of $\text{hypo}(f)$ and $\Pi(f)$. In particular, $C_{\mathbf{m}}$ is convex, and the intersection of any two $C_{\mathbf{m}}$ and $C_{\mathbf{m}'}$ is a face of both.

Proof. Note that we can rewrite

$$P_f(\mathbf{m}, \mathbf{u}) = \Psi_{\text{hypo}(f)}(\mathbf{m}, -1) - (\mathbf{m}, -1) \cdot (\mathbf{u}, f(\mathbf{u})).$$

So for a fixed $\mathbf{m} \in M_{\mathbb{R}}$, the condition $P_f(\mathbf{m}, \mathbf{u}) = 0$ is equivalent to $(\mathbf{m}, -1) \cdot (\mathbf{u}, f(\mathbf{u})) = \Psi_{\text{hypo}(f)}(\mathbf{m}, -1)$, which precisely defines a non-vertical exposed face of $\text{hypo}(f)$. It is clear that all non-vertical faces arise this way.

Therefore, $C_{\mathbf{m}}$ are the projections of non-vertical exposed faces of $\text{hypo}(f)$, and the projection onto $N_{\mathbb{R}}$ induces the claimed bijection. Since faces of convex sets are convex, it follows that $C_{\mathbf{m}}$ is convex.

Moreover, the intersection of $C_{\mathbf{m}}$ and $C_{\mathbf{m}'}$ is the projection of corresponding intersection of non-vertical exposed faces. Since intersection of two exposed faces is again an exposed face, $C_{\mathbf{m}} \cap C_{\mathbf{m}'}$ is a face of both. $\square$

Hence, $\Pi(f)$ is indeed a convex decomposition.

The main result of this section is that Legendre-Fenchel duality of concave functions induces a duality of convex decompositions $\Pi(f) \leftrightarrow \Pi(f^\vee)$ associated to the dual pair, called the *Legendre-Fenchel correspondence*. We first need a definition.

Definition 2.17. Let $E \subset N_{\mathbb{R}}$, $E' \subset M_{\mathbb{R}}$, and Π, Π' be convex decompositions of E, E' , respectively. A bijection $\Pi \rightarrow \Pi', C \mapsto C^*$ is called a *duality of convex decompositions* if

1. for $C, D \in \Pi$, we have $C \subset D \iff D^* \subset C^*$;
2. for all $C \in \Pi$, the sets C and C^* are contained in orthogonal affine subspaces of $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, respectively.

Definition 2.18. Let f be a closed concave function on $N_{\mathbb{R}}$. The *Legendre-Fenchel correspondence* of f is defined as

$$\mathcal{L}f: \Pi(f) \rightarrow \Pi(f^{\vee}), \quad C \mapsto \bigcap_{u \in C} \partial f(u).$$

For this to be well-defined, we need to know $\bigcap_{u \in C} \partial f(u)$ is in $\Pi(f^{\vee})$. This is provided by the following

Lemma 2.19. Let f be a closed concave function on $N_{\mathbb{R}}$, and suppose $C \in \Pi(f)$. Then

$$\bigcap_{u \in C} \partial f(u) = \partial f(u_0)$$

for any $u_0 \in \text{relint}(C)$.

Theorem 2.20. The Legendre-Fenchel correspondence $\mathcal{L}f$ is a duality of convex decompositions, with the inverse given by $\mathcal{L}f^{\vee}$.

Example 2.21. Taking $f = \iota_P$ to be the indicator function of a polytope $P \subset M_{\mathbb{R}}$, we recover the polytope-normal fan duality discussed at the beginning:

Denote by $\Psi_P = f^{\vee}$ the support function of P .

First, we have that $\Pi(f) = \Pi(\iota_P)$ is simply the collection of faces of P . One easy way to see this is by definition:

$$C_m = \partial \Psi_P(m) = \{u \in N_{\mathbb{R}} \mid \iota_P(u) = 0 = (m, u) - \Psi_P(m)\}$$

is the face determined by the vector m .

Next, we can see that $\Pi(\Psi_P) = \Sigma_P$ by considering the regions on which Ψ_P is affine, which are none other than the normal cones.

Finally, by the definition of $\mathcal{L}\iota_P$, we see that for a face $F \subset P$,

$$\begin{aligned} \mathcal{L}\iota_P(F) &= \bigcap_{u \in F} \partial \iota_P(u) = \bigcap_{u \in F} \{m \in M_{\mathbb{R}} \mid (m, v - u) \geq 0 \quad \forall v \in P\} \\ &= \angle(F, P)^{\vee} = \sigma_F \end{aligned}$$

which precisely sends F to its normal cone σ_F . The last equality is by Lemma 2.6.

◇

More generally, whenever f is piecewise affine, the Legendre-Fenchel correspondence can be upgraded to a *duality of polyhedral complexes*.

Definition 2.22. Let P, P' be polyhedra in $N_{\mathbb{R}}, M_{\mathbb{R}}$, respectively. Let Π, Π' be polyhedral complexes in P, P' , respectively. A bijection $\Pi \rightarrow \Pi', C \mapsto C^*$ is a *duality of polyhedral complexes* if

1. for $C, D \in \Pi$, we have $C \subset D \iff D^* \subset C^*$;
2. for all $C, D \in \Pi$, if $C \subset D$, then $\angle(C, D)^{\vee} = \angle(D^*, C^*)$.

Note that a duality of polyhedral complexes is in particular a duality of convex decompositions. Indeed, by considering $\angle(C, C)^\vee = \angle(C^*, C^*)$, we have that C and C^* are orthogonal. Moreover, $\dim C + \dim C^* = n$.

Theorem 2.23. Let f be a piecewise affine concave function with $P = \text{dom}(f)$, $P' = \text{stab}(f)$. Then $\Pi(f), \Pi(f^\vee)$ are polyhedral complexes in P, P' , respectively; and $\mathcal{L}f: \Pi(f) \rightarrow \Pi(f^\vee)$ is a duality of polyhedral complexes. In particular, vertices of $\Pi(f)$ are in bijection with top-dimensional polyhedra in $\Pi(f^\vee)$.

2.5 Monge-Ampère measures

We include a brief discussion of Monge-Ampère measures of concave functions, which are needed in later chapters when we transport height computations to $N_{\mathbb{R}}$. In the case of piecewise affine concave functions, we will obtain a very elegant description of Monge-Ampère measures using the Legendre-Fenchel correspondence.

Let μ be the measure on $M_{\mathbb{R}}$ associated to the differential form $dx_1 \wedge \cdots \wedge dx_n$ for some choice of linear coordinates $(x_1, \dots, x_n)$ on $M_{\mathbb{R}}$. When $(x_1, \dots, x_n)$ are the coordinates associated to an *integral* basis of M , we write $\mu = \text{vol}_M$. Note that any two integral bases differ by a unimodular change of bases, so vol_M does not depend on the choice of basis; indeed, it is the unique Haar measure on $M_{\mathbb{R}}$ where M has covolume 1.

Definition 2.24. Let f be a closed concave function on $N_{\mathbb{R}}$. We define the *Monge-Ampère measure* $\mathcal{M}_\mu(f)$ of f with respect to μ to be such that

$$\mathcal{M}_\mu(f)(E) = \mu(\partial f(E))$$

for Borel sets $E \subset N_{\mathbb{R}}$. When $\mu = \text{vol}_M$, we also write $\mathcal{M}_M(f)$.

We sometimes write simply $\mathcal{M}(f)$ when the measure μ is clear from context.

Since $\text{relint}(\text{stab}(f)) \subset \text{im}(\partial f) \subset \text{stab}(f)$, the total mass of $\mathcal{M}(f)$ is

$$\mathcal{M}(f)(N_{\mathbb{R}}) = \mu(\text{stab}(f)).$$

So $\mathcal{M}(f)$ is a bounded measure when $\text{stab}(f)$ is bounded.

We obtain a very simple description of $\mathcal{M}(f)$ when f is piecewise affine.

Proposition 2.25. Let f be a piecewise affine closed concave function. Then $\mathcal{M}(f)$ is a discrete measure supported on the vertices of $\Pi(f)$.

Proof. Consider any point $x \in N_{\mathbb{R}}$. Recall that by Lemma 2.19, $\partial f(x) = \mathcal{L}f(C)$ for any $C \in \Pi(f)$ such that $x \in \text{relint}(C)$. But by Theorem 2.23, $\Pi(f^\vee)$ is a polyhedral complex, and $\mu(\mathcal{L}f(C)) \neq 0$ if and only if $\mathcal{L}f(C)$ is a top-dimensional polyhedron, which occurs if and only if C is a vertex. Therefore, $\mathcal{M}(f)$ is supported on vertices. $\square$

Finally, we include the following Stokes-type formula relating Monge-Ampère measures and Legendre-Fenchel duals. This formula is needed later in the inductive proof of the local toric height formula – the central result of this exposition.

Theorem 2.26. Let $f: N_{\mathbb{R}} \rightarrow \mathbb{R}$ be a concave function such that $D = \text{stab}(f)$ is compact. Set $\partial D = D \setminus \text{relint}(D)$, and

$$\lambda = \sum_{i=1}^n (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

Then

$$-\int_{N_{\mathbb{R}}} f \, d\mathcal{M}_{\mu}(f) = (n+1) \int_D f^{\vee} \, d\mu - \int_{\partial D} f^{\vee} \lambda.$$

Chapter 3

General Height Theory

Let $\mathbb{K}$ be a number field, and $\mathcal{O}_{\mathbb{K}}$ be its ring of integers. Let X be a variety over $\mathbb{K}$.

In this chapter, we give a rapid introduction to the adelic formulation of Arakelov heights. In the Introduction (Chapter 1), we already discussed some motivations for height functions and a brief history of Arakelov heights. Recall from there that we will take the adelic viewpoint towards Arakelov heights in this exposition, which is the more natural perspective for toric varieties. In particular, we will not work with Gillet-Soulé setup of the pair $(\mathcal{X}, X(\mathbb{C}))$, where $p: \mathcal{X} \rightarrow \text{Spec } \mathcal{O}_{\mathbb{K}}$ is an integral model, together with arithmetic line bundles $\bar{L}$ on X . However, in Remark 3.25, we will discuss how to translate this data into the adelic formalism.

Let $\mathfrak{M}_{\mathbb{K}}$ be the set of places of $\mathbb{K}$, i.e. the set of equivalence classes of non-trivial absolute values on $\mathbb{K}$. We would like to define heights on varieties over $\mathbb{K}$ by summing up individual contributions, called *local heights*, from each place $v \in \mathfrak{M}_{\mathbb{K}}$.

For each place v , denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$ with respect to $|\cdot|_v$. By Ostrowski's Theorem for number fields, the places of $\mathbb{K}$ fall into two types:

1. *archimedean* places: these correspond to real embeddings $\sigma: \mathbb{K} \rightarrow \mathbb{R}$ or complex embeddings $\sigma: \mathbb{K} \rightarrow \mathbb{C}$, and the absolute value is the standard one on $\mathbb{R}$ or $\mathbb{C}$ composed with the embedding. In this case, $\mathbb{K}_v$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.
2. *non-archimedean* places: these are p -adic absolute values induced by the discrete valuations corresponding to prime ideals $\mathfrak{p} \subset \mathcal{O}_{\mathbb{K}}$. In this case, $\mathbb{K}_v$ is a non-archimedean field with a discrete absolute value.

When $\mathbb{K} = \mathbb{Q}$, Ostrowski's theorem says exactly that the places are the usual real absolute value together with the p -adic absolute values.

For each $v \in \mathfrak{M}_{\mathbb{K}}$, write

$$X_v = X \times_{\text{Spec } \mathbb{K}} \text{Spec } \mathbb{K}_v.$$

Each X_v has an associated analytic space. If v is archimedean, this is the usual complex analytic space, with the mild extra convention that over $\mathbb{R}$ one remembers the anti-linear involution induced by complex conjugation (various compatibilities with this involution when working over $\mathbb{R}$ present no substantial obstacles for our exposition and will henceforth be quietly suppressed). If v is non-archimedean, then the associated analytic space is the Berkovich analytification. We denote the resulting analytic space by X_v^{an} .

Interlude. In this exposition, we will for the most part treat Berkovich analytic spaces as black boxes, thinking of them simply as non-archimedean analogues of complex analytic spaces. For flavour, we now give a very brief description of Berkovich spaces defined in [Ber90].

Let K be a non-archimedean field equipped with a non-archimedean absolute value $|\cdot|$. Let A be a finitely generated K -algebra. Then the *Berkovich spectrum* $\mathcal{M}(A)$ is defined to be the set of multiplicative seminorms on A extending the absolute value $|\cdot|$ on K (a seminorm is like a norm except it may have a non-trivial kernel). This should be reminiscent of the construction of the algebraic spectrum $\text{Spec } A$. Moreover, there is a map $\mathcal{M}(A) \rightarrow \text{Spec } A$.

The set $\mathcal{M}(A)$ can be endowed with a topology and a sheaf of analytic functions making it into a locally ringed space. Given a scheme X of finite type over K , the local constructions $\mathcal{M}(A)$ glue to a locally ringed space, which we call X^{an} . The above map glues to a continuous map $X^{\text{an}} \rightarrow X$.

Similarly, if L is an invertible sheaf on X , then its base change L_v to X_v induces an analytic line bundle L_v^{an} on X_v^{an} . The local height at v will be constructed by equipping such analytic line bundles with metrics and then integrating certain functions against measures associated to these metrics.

Since the local constructions are independent of the original number field once the completion $\mathbb{K}_v$ has been chosen, we now temporarily leave the global setting. Resetting notation, let K be a field complete with respect to an absolute value $|\cdot|$, let X be a variety over K , and let L be an invertible sheaf on X . The reader should keep in mind the motivating example

$$K = \mathbb{K}_v, \quad X = X_v, \quad L = L_v$$

coming from a variety and a line bundle over the number field $\mathbb{K}$.

As above, the associated analytic space X^{an} is either the complex analytic space or the Berkovich analytic space depending on whether K is archimedean or non-archimedean. In both cases, the algebraic line bundle L on X induces an analytic line bundle L^{an} on X^{an} .

We can now define (*continuous*) *metrics* on L^{an} , regardless of whether K is archimedean or non-archimedean.

Definition 3.1. A (*continuous*) *metric* on L^{an} is an assignment that produces a

continuous function

$$\|s(\cdot)\| : \mathcal{U} \rightarrow \mathbb{R}_{\geq 0}$$

for any open set $\mathcal{U} \subset X^{\text{an}}$ and any section s of L^{an} over $\mathcal{U}$, such that

1. these assignments commute with restrictions: i.e., if $V \subset \mathcal{U}$, then $\|(s|_V)(\cdot)\|$ is the restriction of $\|s(\cdot)\|$ to V ;
2. for all $p \in \mathcal{U}$, $\|s(p)\| = 0$ if and only if $s(p) = 0$;
3. for all $p \in \mathcal{U}$ and regular functions f on $\mathcal{U}$, $\|(fs)(p)\| = |f(p)| \|s(p)\|$.

We denote by $\bar{L}$ the pair $(L, \|\cdot\|)$.

Note that $|f(p)|$ is simply the usual absolute value of complex numbers if $K = \mathbb{C}$, while the reader should believe that there is similarly a way to assign such absolute values on Berkovich analytic spaces.

Remark 3.2. Definition 3.1 might strike one as a strange way to define a metric, but it is a convenient way to package continuity. Note that the above definition in particular induces a norm-like assignment

$$\|\cdot\|_x : L_x^{\text{an}} \rightarrow \mathbb{R}_{\geq 0}$$

for any fibre L_x^{an} , defined as follows: for any $v \in L_x^{\text{an}}$, take a local section s such that $s(x) = v$, and define $\|v\|_x$ to be $\|s(x)\|$. This is well-defined because if we have any other $s'(x) = v$, then there is a regular function f near x such that $f(x) = 1$ and $s' = fs$, and so $\|s(x)\| = \|s'(x)\|$ by property (3) in the definition. ∇

We next refine the notion of metrised line bundles separately in the archimedean and non-archimedean cases.

3.1 Smooth metrics and associated measures

We begin to explore the situation more in the archimedean case. Let X be a variety over $\mathbb{C}$, and L an invertible sheaf on X . Let X^{an} and L^{an} be as above.

A metric on L^{an} as in Definition 3.1 is *smooth* if the function $\|s(\cdot)\|^2$ is smooth for any local section s of L^{an} .

Example 3.3. Let $X = \mathbb{P}_{\mathbb{C}}^n$ be the complex projective space, with homogeneous coordinates $[x_0 : \dots : x_n]$. Let $L = \mathcal{O}(1)$. Recall that a global section $s \in H^0(X, L)$ can be identified with homogeneous linear polynomials $f(x_0, \dots, x_n)$. We define the *Fubini-Study metric* to be such that

$$\|s(x)\| = \frac{|f(x_0, \dots, x_n)|}{(\sum_{i=0}^n |x_i|^2)^{1/2}}$$

at a point $x = [x_0 : \dots : x_n]$.

Notice that the right-hand side is invariant under scaling, hence well-defined. This definition extends uniquely to all rational sections of $\mathcal{O}(1)$, by the second requirement in Definition 3.1. This metric is denoted by $\|\cdot\|_{\text{FS}}$. This is a smooth metric. It is also semipositive, as defined below. $\diamond$

Remark 3.4. This definition may seem odd at first, but it is in fact rather canonical. For example, it is the unique Hermitian metric on $\mathcal{O}(1)$ invariant under the unitary group $U(n+1)$ (see [Huy05, Section 3.1]). ∇

Now let $\bar{L}$ be a smooth metrised line bundle. We can define a global closed $(1,1)$ -form as follows: let s be a local non-vanishing section of L^{an} on an open set $U \subset X^{\text{an}}$, and define locally

$$c_1(\bar{L}) = \partial\bar{\partial} \log \|s\|^2.$$

This does not depend on the choice of sections s , and hence can be defined globally on X^{an} . We call $c_1(\bar{L})$ the *first Chern form* of $\bar{L}$. The corresponding cohomology class does *not* depend on the metric.

We next define the notion of *semipositive metrics*, which are needed to define local heights.

Definition 3.5. Let $\bar{L}$ be a smooth metrised line bundle, and $D \subset \mathbb{C}$ be the closed unit disk. Then $\bar{L}$ is *semipositive* if, for every holomorphic map $\varphi: D \rightarrow X^{\text{an}}$,

$$\frac{1}{2\pi i} \int_D \varphi^* c_1(\bar{L}) \geq 0.$$

Remark 3.6. It is important that the integral is over the *closed* unit disk. If D had no boundary, then, since the cohomology class of $c_1(\bar{L})$ is independent of the metric, the integral would also be independent of the metric by Stokes' Theorem, making the definition useless. ∇

Suppose $\bar{L}_0, \dots, \bar{L}_{d-1}$ are metrised line bundles on X^{an} , and $Y \subset X$ is a d -dimensional irreducible subvariety, we can define a signed measure on X as follows. Suppose δ_Y is the current of integration along Y^{an} , given by

$$\delta_Y(\omega) = \frac{1}{(2\pi i)^d} \int_{Y^{\text{an}}} \omega$$

for complex-valued smooth $2d$ -forms on X^{an} . Then we can define the current

$$c_1(\bar{L}_0) \wedge \dots \wedge c_1(\bar{L}_{d-1}) \wedge \delta_Y$$

which evaluates to

$$\frac{1}{(2\pi i)^d} \int_{Y^{\text{an}}} f|_{Y^{\text{an}}} \cdot (c_1(\bar{L}_0) \wedge \dots \wedge c_1(\bar{L}_{d-1}))_{Y^{\text{an}}}$$

for a smooth complex-valued function f on X^{an} .

So $c_1(\overline{L}_0) \wedge \cdots \wedge c_1(\overline{L}_{d-1}) \wedge \delta_Y$ induces a signed measure on X^{an} . It is a measure if $\overline{L}_0, \dots, \overline{L}_{d-1}$ are all semipositive.

By extending linearly, we have defined a signed measure $c_1(\overline{L}_0) \wedge \cdots \wedge c_1(\overline{L}_{d-1}) \wedge \delta_Y$ given any d -dimensional cycle Y on X , and metrised line bundles $\overline{L}_0, \dots, \overline{L}_{d-1}$ on X^{an} .

3.2 Algebraic metrics and associated measures

The analogue of smooth metrics in the non-archimedean case is algebraic metrics. It is defined in a quite different fashion. But we will similarly study semipositivity and associated measures in parallel to the archimedean case.

Let K be a non-archimedean complete valued field (with a non-trivial absolute value). Let $K^\circ = \{\alpha \in K \mid |\alpha| \leq 1\}$ be the valuation ring of K , and $K^{\circ\circ} = \{\alpha \in K \mid |\alpha| < 1\}$ the maximal ideal of K° . Denote by $k = K^\circ/K^{\circ\circ}$.

For this exposition, we will only need the case when K is discretely valued, in which case K° is a discrete valuation ring (DVR). We will assume this is the case from now on.

Let X be a *proper* variety over K , and L a line bundle on X .

Algebraic metrics on L^{an} are defined using models. Let $S = \text{Spec } K^\circ$. Since K° is a DVR, S has two points. Denote the special point by 0 and the generic point by η . Given a scheme $\mathcal{X} \rightarrow S$, let $\mathcal{X}_0 = \mathcal{X}_k = \mathcal{X} \times_S \text{Spec } k$ be the special fibre, and $\mathcal{X}_\eta = \mathcal{X}_K = \mathcal{X} \times_S \text{Spec } K$ the generic fibre.

Definition 3.7. A *model* over S of X is a scheme $\mathcal{X}$ of finite type and flat over S , together with a chosen isomorphism $X \simeq \mathcal{X}_K$. $\mathcal{X}$ is a *proper model* if $\mathcal{X} \rightarrow S$ is proper.

One can extend this notion to models of line bundles.

Definition 3.8. Let L be an invertible sheaf on X . A *model* over S of the pair (X, L) is a triple $(\mathcal{X}, \mathcal{L}, e)$, where $\mathcal{X}$ is a model of X as above, and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and $e \geq 1$ an integer together with a chosen isomorphism $\mathcal{L}|_{\mathcal{X}} \simeq L^{\otimes e}$. The model $(\mathcal{X}, \mathcal{L}, e)$ is *proper* if $\mathcal{X}$ is proper. When $e = 1$, $(\mathcal{X}, \mathcal{L}, e)$ is shortened to $(\mathcal{X}, \mathcal{L})$.

Remark 3.9. In both definitions above, the isomorphisms are part of the data of the model. ∇

Given a proper model $(\mathcal{X}, \mathcal{L}, e)$ of (X, L) , we can construct a metric on L^{an} , called the *metric* induced by the model $(\mathcal{X}, \mathcal{L}, e)$. Such metrics induced by proper models are called *algebraic metrics*. The model itself is not part of an algebraic metric, and different models may induce the same metric; see [BPS14, Proposition 1.3.6].

The construction is included below for completeness and for flavour. For unexplained terms, refer to [BPS14, Section 1.2] for details.

Construction 3.10. Let $(\mathcal{X}, \mathcal{L}, e)$ be a proper model of (X, L) . Let s be a local section of L^{an} defined around $p \in X^{\text{an}}$. Let $\mathcal{U} \subset \mathcal{X}$ be a trivialising open set for $\mathcal{L}$ containing the image $\text{red}(p)$ of p along the reduction map

$$\text{red}: X^{\text{an}} \rightarrow \mathcal{X}_0.$$

Let σ be a generator of $\mathcal{L}|_{\mathcal{U}}$, and let $U = \mathcal{U} \cap X_{\eta} \subset X$. Then by assumption, there exists a regular function $\lambda \in \mathcal{O}_{U^{\text{an}}}(U^{\text{an}})$ such that $s^{\otimes e} = \lambda \sigma$. Then the metric $\|\cdot\|_{\mathcal{X}, \mathcal{L}, e}$ is defined by

$$\|s(p)\|_{\mathcal{X}, \mathcal{L}, e} = |\lambda(p)|^{1/e}.$$

It can be shown that this does not depend on the choices made.

We next turn to the notion of semipositivity.

Definition 3.11. Let (X, L) be as before. A proper model $(\mathcal{X}, \mathcal{L}, e)$ is said to be a *semipositive model* if for every curve $C \subset \mathcal{X}$ contained entire in the special fibre $\mathcal{X}_0$, we have $\deg_{\mathcal{L}} C \geq 0$.

A metrised line bundle $\bar{L}$ is said to *have a semipositive model* if there exists a semipositive model inducing its metric.

Example 3.12. Let $X = \mathbb{P}_K^n$ and $L = \mathcal{O}(1)$. We will state, but not prove, the algebraic metric given by the model $\mathcal{X} = \mathbb{P}_{K^{\circ}}^n, \mathcal{L} = \mathcal{O}_{\mathbb{P}_{K^{\circ}}^n}(1), e = 1$. Given a section $s \in H^0(X, L)$ represented by a homogeneous linear polynomial $f(x_0, \dots, x_n)$, the model induces the *canonical metric*, given by

$$\|s(x)\| = \frac{|f(x_0, \dots, x_n)|}{\max_i |x_i|}$$

at a point $x = [x_0 : \dots : x_n] \in (\mathbb{P}_K^n)^{\text{an}}$ (I have not defined for you what these absolute values are in the non-archimedean setting). This metric is denoted by $\|\cdot\|_{\text{can}}$.

The model turns out to be semipositive, essentially because $\mathcal{O}(1)$ is ample.

◇

Finally, given a sequence of metrised line bundles with algebraic metrics and a cycle, we can again associate a signed measure on X^{an} .

Definition 3.13. Suppose $\bar{L}_0, \dots, \bar{L}_{d-1}$ are metrised line bundles with algebraic metrics on X^{an} , and Y is a d -dimensional cycle on X . Then we can again define a signed measure on X^{an} , denoted by

$$c_1(\bar{L}_0) \wedge \dots \wedge c_1(\bar{L}_{d-1}) \wedge \delta_Y.$$

The notation is purely formal and to emphasise the parallel with the archimedean case; we are not claiming the existence of differential forms or currents in the present setting.

We omit the specific construction and refer the reader to [BPS14, Definition 1.3.15]. We remark that this measure is a discrete measure, supported on various points to be thought of as generic points of some sort which are related to Y .

3.3 Local heights from metrics

In this section, K is either a complete archimedean field or a discretely valued complete non-archimedean field, as considered in the previous two sections, and X is a proper variety over K , and L is a line bundle on X .

Given two metrics on L^{an} , it is not hard to define a distance between them (see [BPS14, Section 1.4]), so we can speak of convergence of metrics on L^{an} .

We next define *semipositive metrics* on L^{an} via a limiting process.

Definition 3.14. A metric $\|\cdot\|$ is said to be *semipositive* if there exists a sequence $(\|\cdot\|_l)_{l \in \mathbb{N}}$ of semipositive smooth metrics (if K is archimedean) or algebraic metrics with semipositive models (if K is non-archimedean) on L^{an} such that $\|\cdot\|_l \rightarrow \|\cdot\|$.

If $\|\cdot\|$ is semipositive, we also say that $\bar{L}$ is semipositive. If there are semipositive line bundles $\bar{M}, \bar{N}$ such that $\bar{L} = \bar{M} \otimes \bar{N}^{-1}$, then we say $\bar{L}$ is *DSP* (for “difference of semipositive”). In the literature, DSP is also called *admissible*.

Note that there is no reason for the limit of smooth or algebraic metrics to be again smooth or algebraic, so this is genuinely a new definition. If a metric is smooth, then the present definition of semipositivity agrees with the original definition. For algebraic metrics, having semipositive model trivially implies semipositive; but conversely, it is not known whether a semipositive metric which happens to be algebraic has a semipositive model.

Example 3.15. We have seen that $\|\cdot\|_{\text{can}}$ is an algebraic metric with a semipositive model on $\mathbb{P}_K^n$ when K is non-archimedean. The same formula

$$\|s(x)\| = \frac{|f(x_0, \dots, x_n)|}{\max_i |x_i|}$$

also defines a metric on $\mathbb{P}_{\mathbb{C}}^n$, also called the *canonical metric*. In this case, this metric is not smooth, but it is the limit of semipositive smooth metrics $\|\cdot\|_m$ given by

$$\|s(x)\|_m = \frac{|f(x_0, \dots, x_n)|}{(\sum_{i=0}^n |x_i|^{2m})^{1/2m}}.$$

◇

The measures associated to smooth metrised line bundles or metrised line bundles with semipositive models, considered in the previous sections, behave

well with respect to the limiting process ([BPS14, Proposition 1.4.5]). This gives us the following.

Definition 3.16. Suppose $\overline{L}_0, \dots, \overline{L}_{d-1}$ are semipositive metrised line bundles on X^{an} , and Y is a d -dimensional cycle on X . Then we can define a signed measure on X^{an} , denoted by

$$c_1(\overline{L}_0) \wedge \dots \wedge c_1(\overline{L}_{d-1}) \wedge \delta_Y.$$

This definition extends to DSP line bundles multilinearly.

Local heights are defined inductively on dimensions. We need the following notion of transversality.

Definition 3.17. Let Y be a d -dimensional cycle of X , and $(L_i, s_i)_{0 \leq i \leq d}$ be a collection of line bundles with rational sections. Then $s_0, \dots, s_d$ are said to *meet Y properly* if, for any $I \subset \{0, \dots, d\}$,

$$Y \cap \bigcap_{i \in I} |\text{div}(s_i)|$$

has pure dimension $d - |I|$ (by convention, $\dim \emptyset = -1$).

Now we are ready to define local heights.

Definition 3.18. The *local height* on X is a function that associates a real number $h_{\overline{L}_0, \dots, \overline{L}_d}(Y; s_0, \dots, s_d)$ to every d -cycle Y on X and each collection $(L_i, s_i)_{0 \leq i \leq d}$ of DSP line bundles with rational sections meeting Y properly, defined inductively on d as follows:

1. $h_{\overline{L}_0}(\emptyset) = 0$
2. if $d \geq 0$, then

$$\begin{aligned} h_{\overline{L}_0, \dots, \overline{L}_d}(Y; s_0, \dots, s_d) &= h_{\overline{L}_0, \dots, \overline{L}_{d-1}}(Y \cdot \text{div}(s_d); s_0, \dots, s_{d-1}) \\ &\quad - \int_{X^{\text{an}}} \log \|s_d\| c_1(\overline{L}_0) \wedge \dots \wedge c_1(\overline{L}_{d-1}) \wedge \delta_Y. \end{aligned}$$

It is very useful to keep in mind that for a single pair $(\overline{L}, s)$ and $p \in X$, the height of p is simply

$$h_{\overline{L}}(p; s) = -\log \|s(p)\|.$$

A few remarks about this definition are in order. Firstly, note that by the definition of proper intersections above, the condition is indeed satisfied for the inductive case: i.e., we indeed have $s_0, \dots, s_{d-1}$ meeting $Y \cdot \text{div}(s_d)$. Here, $Y \cdot \text{div}(s_d)$ is the $(d-1)$ -cycle formed by taking the divisor of $s_d|_Y$, which is well-defined because s_d meets Y properly.

Secondly, the integral is indeed well-defined with respect to the given measure, despite $\log \|s_d\|$ having some logarithmic poles along $\text{div}(s_d)$. This is

not trivial; see [BPS14, Theorem 1.4.10] for details. We remark that $\text{div}(s_d)$ has measure zero, as one may reasonably expect from the proper intersection condition.

Thirdly, if Y is projective, then sections $s_0, \dots, s_d$ intersecting Y properly can always be found, due to the Moving Lemma from intersection theory.

Finally, sometimes we may only care about the height of a closed point $x \in X$, given (L_0, s_0) . Then the simplest case of the above definition applies, and we have that as long as $x \notin |\text{div}(s_0)|$,

$$h_{L_0}^-(x; s_0) = -\log \|s_0(x)\|.$$

3.4 Global heights

We now return to the case of defining *global heights* of a variety over $\mathbb{Q}$ by synthesising the inputs from local heights. While we keep this case over $\mathbb{Q}$ in mind, the correct generality to work in for our current setup is *adelic fields* and can be stated without much extra effort.

Let $\mathbb{K}$ be a field and $\mathfrak{M}$ a collection of pairs $(|\cdot|_v, n_v)$, where $|\cdot|_v$ is an absolute value on $\mathbb{K}$ and $n_v \in \mathbb{R}_{>0}$ is called the *weight* of the absolute value. Such a pair $(|\cdot|_v, n_v) \in \mathfrak{M}$ is called a *place* and denoted by v .

Definition 3.19. The pair $(\mathbb{K}, \mathfrak{M})$ is called an *adelic field* if

1. Every absolute value $|\cdot|$ in $\mathfrak{M}$ is either archimedean or associated to a non-trivial discrete valuation (hence non-archimedean);
2. for each $\alpha \in \mathbb{K}^\times$, $|\alpha|_v = 1$ except for a finite number of places v .

We denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$ with respect to $|\cdot|_v$. As in the number field case discussed at the start of this chapter, $\mathbb{K}_v$ is $\mathbb{R}$, $\mathbb{C}$, or a non-archimedean discretely valued field.

We define the *defect map* to be the group homomorphism $\text{def}: \mathbb{K}^\times \rightarrow \mathbb{R}$ given by

$$\text{def}(\alpha) = \sum_{v \in \mathfrak{M}} n_v \log |\alpha|_v.$$

The subgroup $\text{def}(\mathbb{K}^\times) \subset \mathbb{R}$ is the ambiguity global heights will be defined up to, and we say $\mathbb{R}/\text{def}(\mathbb{K}^\times)$ is the *group of global heights* of $\mathbb{K}$. If $\text{def}(\alpha) = 0$ for all $\alpha \in \mathbb{K}$, we say that $\mathbb{K}$ satisfies the product formula. In this case, global heights will be real numbers.

Let $\mathfrak{M}_{\mathbb{Q}}$ be the collection of the real absolute value and the p -adic absolute values (where as usual $|p|_p = p^{-1}$), with all $n_v = 1$. Then $(\mathbb{Q}, \mathfrak{M}_{\mathbb{Q}})$ is an adelic field satisfying the product formula.

The following definition makes finite extensions of adelic fields into adelic fields.

Definition 3.20. Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field, and $\mathbb{F}/\mathbb{K}$ a finite extension. Let $\mathfrak{M}_{\mathbb{F}}$ be the set of absolute values $|\cdot|_w$ which extend $|\cdot|_v$ for some $v \in \mathfrak{M}$. We set

$$n_w = \frac{[\mathbb{F}_w : \mathbb{K}_v]}{[\mathbb{F} : \mathbb{K}]} n_v.$$

Then $(\mathbb{F}, \mathfrak{M}_{\mathbb{F}})$ is an adelic field, called an *adelic field extension* of $(\mathbb{K}, \mathfrak{M})$.

Adelic field extensions preserve the product formula. In particular, all number fields (finite extensions of $\mathbb{Q}$) are adelic fields satisfying the product formula. For any smooth projective curve C over a field k , there is also a way of making its function field $K(C)$ and any finite extension of $K(C)$ into adelic fields which satisfy the product formula.

Definition 3.21. Number fields and finite extensions of $K(C)$ as in the last paragraph are called *global fields*. In particular, global fields satisfy the product formula.

We will now define global heights. To simplify the definition, let $(\mathbb{K}, \mathfrak{M})$ be an adelic field, and let X be a *projective* variety over $\mathbb{K}$, and L a line bundle on X . More generally, this can be done for proper varieties X , by instead passing to birational projective models of X or of the subvarieties $Y \subset X$ considered below.

For each $v \in \mathfrak{M}$, let $X_v = X \times_{\mathbb{K}} \text{Spec } \mathbb{K}_v$, and let L_v be the pullback of L to X_v . A *metric* on L is a choice of metric $\|\cdot\|_v$ on L_v^{an} for each $v \in \mathfrak{M}$. Denote by $\bar{L} = (L, (\|\cdot\|_v)_v)$ the pair L together with this family of metrics. $\bar{L}$ is called *semipositive* (resp. *DSP*) if all $\|\cdot\|_v$ are semipositive (resp. *DSP*).

Let Y be a d -dimensional cycle on X , and $(L_i, s_i)_{0 \leq i \leq d}$ *DSP* line bundles on X with rational sections meeting Y properly. Since we assumed X to be projective, such sections always exist. We write

$$h_{v, \bar{L}_0, \dots, \bar{L}_d}(Y; s_0, \dots, s_d) = h_{\bar{L}_{0,v}, \dots, \bar{L}_{d,v}}(Y_v; s_{0,v}, \dots, s_{d,v}),$$

where the right-hand side is the local height defined in the last section with rational sections $s_{i,v}$ induced from s_i .

We would like to define the global height $h_{\bar{L}_0, \dots, \bar{L}_d}(Y; s_0, \dots, s_d)$ by

$$\sum_{v \in \mathfrak{M}} n_v h_{v, \bar{L}_0, \dots, \bar{L}_d}(Y; s_0, \dots, s_d). \quad (*)$$

But a priori, this is not a finite sum, hence not well-defined. But in the following case, it is well-defined.

Definition 3.22. Let X, Y be as above. We say that Y is *integrable* with respect to $\bar{L}_0, \dots, \bar{L}_d$ if there exists rational sections s_i of L_i meeting Y properly such that

$$h_{v, \bar{L}_0, \dots, \bar{L}_d}(Y; s_0, \dots, s_d) = 0$$

for all but finitely many $v \in \mathfrak{M}$.

Next, we show that integrability can be detected with any choice of sections s_i meeting Y properly.

Proposition 3.23. Given rational sections s_i of L_i meeting Y properly such that

$$h_{v, \overline{L_0}, \dots, \overline{L_d}}(Y; s_0, \dots, s_d) = 0$$

for all but finitely many $v \in \mathfrak{M}$, then the same holds for any other rational sections s'_i of L_i meeting Y properly.

Proof. By [BPS14, Theorem 1.4.17, (3)], there exists a constant β in $\mathbb{K}^\times$ not depending on v such that

$$h_{v, \overline{L_0}, \dots, \overline{L_d}}(Y; s_0, \dots, s_d) - h_{v, \overline{L_0}, \dots, \overline{L_d}}(Y; s'_0, \dots, s'_d) = \log|\beta|_v.$$

Now, by the second axiom of adelic fields, $\log|\beta|_v = 0$ for all but finitely many places v . The result follows. $\square$

Now suppose Y is an integrable d -dimensional cycle. Then we may define $h_{\overline{L_0}, \dots, \overline{L_d}}(Y; s_0, \dots, s_d)$ by $(*)$ above, which is now a finite sum. Furthermore, the above proof shows that for any other sections s'_d meeting Y properly, there is $\beta \in \mathbb{K}^\times$

$$\begin{aligned} h_{\overline{L_0}, \dots, \overline{L_d}}(Y; s_0, \dots, s_d) - h_{\overline{L_0}, \dots, \overline{L_d}}(Y; s'_0, \dots, s'_d) \\ = \sum_{v \in \mathfrak{M}} n_v \log|\beta|_v \in \text{def}(\mathbb{K}^\times). \end{aligned}$$

Definition 3.24. For an integrable d -cycle Y , we can thus define its *global height* $h_{\overline{L_0}, \dots, \overline{L_d}}(Y)$ to be the class of any $h_{\overline{L_0}, \dots, \overline{L_d}}(Y; s_0, \dots, s_d)$ in $\mathbb{R}/\text{def}(\mathbb{K}^\times)$.

In particular, if $(\mathbb{K}, \mathfrak{M})$ satisfies the product formula (for example, if $\mathbb{K}$ is a global field), then $h_{\overline{L_0}, \dots, \overline{L_d}}(Y)$ is a well-defined real number.

So far, we have imposed no compatibility conditions on the family of metrics $\|\cdot\|_{i,v}$ on $L_{i,v}$ at different places. We finish the review of height theory by discussing a compatibility condition which makes all cycles integrable.

Let $(\mathbb{K}, \mathfrak{M})$ be a global field. Let $S \subset \mathfrak{M}$ be a finite subset including all the archimedean places. Denote by $\mathbb{K}_S^\circ$ the subring of $\mathbb{K}$ consisting of elements $\alpha \in \mathbb{K}$ for which $|\alpha|_v \leq 1$ for all $v \notin S$. Note that $\mathbb{K}_S^\circ \subset \mathbb{K}_v^\circ$ for all $v \notin S$.

Let X be a proper variety over $\mathbb{K}$ with a line bundle L . Recall that X_v and L_v denote the localisation of X and L at the place v . We say that a metric on L is *quasi-algebraic* if there exists a finite subset $S \subset \mathfrak{M}$ including all archimedean places and a proper model $(\mathcal{X}, \mathcal{L}, e)$ of (X, L) over $\mathbb{K}_S^\circ$ such that the metric $\|\cdot\|_v$ on L_v is induced by the localisation of this model for every $v \notin S$. Here, a proper model $(\mathcal{X}, \mathcal{L}, e)$ is defined in an analogous fashion as in Definitions 3.7 and 3.8. Note that the pullback $(\mathcal{X} \times_{\text{Spec } \mathbb{K}_S^\circ} \text{Spec } \mathbb{K}_v^\circ, \mathcal{L} \times_{\text{Spec } \mathbb{K}_S^\circ} \text{Spec } \mathbb{K}_v^\circ, e)$ is a proper model of (X_v, L_v) for all $v \notin S$, in the sense of Definition 3.8.

Remark 3.25. If S contains only the archimedean places, and $e = 1$, then we recover the Gillet-Soulé setup. In other words, given a proper model $p: \mathcal{X} \rightarrow \operatorname{Spec} \mathcal{O}_K$ of X and arithmetic line bundles $\bar{L}$ on X , we can define global heights using the current formalism. ∇

Global heights with respect to quasi-algebraic metrics over global fields are well-defined real numbers for all cycles.

Theorem 3.26. Let $\mathbb{K}$ be a global field and X a proper variety over $\mathbb{K}$ of dimension n . Then for all $d \leq n$ and metrised line bundles $\bar{L}_0, \dots, \bar{L}_d$ with quasi-algebraic metrics, any d -dimensional cycle Y on X is integrable with respect to $\bar{L}_0, \dots, \bar{L}_d$.

Example 3.27. We put this theorem and the above remark into context by returning to the guiding example of $\mathbb{P}_{\mathbb{Q}}^n$. Let $L = \mathcal{O}(1)$. To define a global height on $\mathbb{P}_{\mathbb{Q}}^n$, choose S to contain only the place at ∞ . Then $\mathbb{Q}_S^\circ = \mathbb{Z}$. Then $(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(1), e = 1)$ is a proper model for $\mathbb{P}_{\mathbb{Q}}^n$ over $\mathbb{Z}$; this determines algebraic metrics at all the non-archimedean places, which are the canonical metrics defined in Example 3.12. We choose the Fubini-Study metric at ∞ (Example 3.3). This data together makes L into a metrised line bundle on $\mathbb{P}_{\mathbb{Q}}^n$. Therefore, the above theorem tells us that we can define the heights of any cycle on $\mathbb{P}_{\mathbb{Q}}^n$. We shall return to the toric computation of this global height in Chapter 5. $\diamond$

Chapter 4

Toric Metrics

In this chapter, we begin to explore metrised line bundles on toric varieties.

Let Σ be a fan, and X_Σ be the associated toric variety. Let $L = \mathcal{O}(D_\Psi)$ be a toric line bundle equipped with a metric $\|\cdot\|$ on L^{an} . Our goal in this section is to define and study an appropriate notion of what it means for $\|\cdot\|$ to be torus-invariant.

As we will discuss more later, we will require the metric $\|\cdot\|$ to be invariant not under the action of the entire analytic torus $\mathbb{T}^{\text{an}}$, but under the action of the *compact torus*, defined in both archimedean and non-archimedean cases as

$$S = \{p \in \mathbb{T}^{\text{an}} \mid |x^m(p)| = 1 \quad \forall m \in M\}.$$

In the archimedean case, the analytic torus $\mathbb{T}^{\text{an}}$ is simply $(\mathbb{C}^\times)^n$, and S is simply the set of points whose coordinates all have absolute values equal to 1.

Therefore, we will be mainly concerned with S -actions throughout. This leads us to consider the variety with corners associated to Σ , denoted by $X_\Sigma(\mathbb{R}_{\geq 0})$. In the archimedean case, this turns out to be precisely the set of S -orbits of X^{an} . We will find it very fruitful to characterise metrics on toric varieties through $X_\Sigma(\mathbb{R}_{\geq 0})$.

4.1 The variety with corners

Recall that the $\mathbb{C}$ -points of the affine toric variety U_σ associated to a cone $\sigma \in \Sigma$ can be described as

$$U_\sigma(\mathbb{C}) = \text{Hom}_{\text{sg}}(M_\sigma, (\mathbb{C}, \times)),$$

where $M_\sigma = \sigma^\vee \cap M$.

Similarly, we now define

$$U_\sigma(\mathbb{R}_{\geq 0}) = \text{Hom}_{\text{sg}}(M_\sigma, (\mathbb{R}_{\geq 0}, \times)),$$

where mapping into $\mathbb{R}_{\geq 0}$ gives it a natural topology inherited from the Euclidean topology on $\mathbb{R}_{\geq 0}$. Equivalently, it is the coarsest topology which makes

all evaluation maps at $m \in M_\sigma$ continuous, that is, the maps $\gamma \mapsto \gamma(m)$ for $\gamma \in \mathcal{U}_\sigma(\mathbb{R}_{\geq 0})$.

The same proof in the algebraic case shows that the inclusion of a face $\tau \subset \sigma$ induces the inclusion $\mathcal{U}_\tau(\mathbb{R}_{\geq 0}) \subset \mathcal{U}_\sigma(\mathbb{R}_{\geq 0})$ as an open set. Therefore, the topological spaces $\mathcal{U}_\sigma(\mathbb{R}_{\geq 0})$ glue to define a topological space, which we define to be the *variety with corners* $X_\Sigma(\mathbb{R}_{\geq 0})$ associated to Σ .

Similar to the algebraic case where $\mathcal{U}_0 = \mathbb{T}$ associated to the cone at the origin is the torus, in this case we have

$$\mathbb{T}(\mathbb{R}_{\geq 0}) := \mathcal{U}_0(\mathbb{R}_{\geq 0}) = \text{Hom}_{\text{sg}}(M, (\mathbb{R}_{\geq 0}, \times)) \simeq (\mathbb{R}_{> 0})^n.$$

This is a group acting naturally on $X_\Sigma(\mathbb{R}_{\geq 0})$. More explicitly, given $\theta \in \mathbb{T}(\mathbb{R}_{\geq 0})$ and $\gamma \in \mathcal{U}_\sigma(\mathbb{R}_{\geq 0})$, we define $\theta \cdot \gamma \in \mathcal{U}_\sigma(\mathbb{R}_{\geq 0}) = \text{Hom}_{\text{sg}}(M_\sigma, (\mathbb{R}_{\geq 0}, \times))$ by

$$(\theta \cdot \gamma)(m) = \theta(m) \cdot \gamma(m), \quad (4.1)$$

for all $m \in M_\sigma$.

We next define a map ρ_Σ from the analytic toric variety X_Σ^{an} to this variety with corners $X_\Sigma(\mathbb{R}_{\geq 0})$.

In the archimedean case, for each cone $\sigma \in \Sigma$, we have $\mathcal{U}_\sigma^{\text{an}} = \mathcal{U}_\sigma(\mathbb{C}) = \text{Hom}_{\text{sg}}(M_\sigma, (\mathbb{C}, \times))$, and we simply compose with the absolute value to obtain a map

$$\rho_\sigma: \mathcal{U}_\sigma^{\text{an}} = \text{Hom}_{\text{sg}}(M_\sigma, (\mathbb{C}, \times)) \xrightarrow{|\cdot|} \text{Hom}_{\text{sg}}(M_\sigma, (\mathbb{R}_{\geq 0}, \times)) = \mathcal{U}_\sigma(\mathbb{R}_{\geq 0}).$$

In the non-archimedean case with field K , the points p in $\mathcal{U}_\sigma^{\text{an}}$ correspond to multiplicative seminorms on $K[M_\sigma]$. Denote the seminorm corresponding to p by $\|\cdot\|_p$, and recall that we often denote $\|\chi^m\|_p$ by $|\chi^m(p)|$. Then we define $\rho_\sigma(p) \in \mathcal{U}_\sigma(\mathbb{R}_{\geq 0}) = \text{Hom}_{\text{sg}}(M_\sigma, (\mathbb{R}_{\geq 0}, \times))$ to be the semigroup homomorphism sending $\chi^m \mapsto |\chi^m(p)|$.

In both the archimedean and non-archimedean cases, the maps ρ_σ glue to give a continuous map

$$\rho_\Sigma: X_\Sigma^{\text{an}} \rightarrow X_\Sigma(\mathbb{R}_{\geq 0}).$$

We can restrict our attention to the torus and consider the map $\rho_0: \mathbb{T}^{\text{an}} \rightarrow \mathbb{T}(\mathbb{R}_{\geq 0})$. Note that $\mathbb{T}(\mathbb{R}_{\geq 0})$ is naturally isomorphic to $N_{\mathbb{R}}$ via the exponential and logarithmic maps: Recall that $\mathbb{T}(\mathbb{R}_{\geq 0}) = \text{Hom}(M, \mathbb{R}_{\geq 0})$ and $N_{\mathbb{R}} = \text{Hom}(M, \mathbb{R})$. Define $\log: \mathbb{T}(\mathbb{R}_{\geq 0}) \rightarrow N_{\mathbb{R}}$ by

$$\text{Hom}(M, \mathbb{R}_{\geq 0}) \rightarrow \text{Hom}(M, \mathbb{R}), \quad \varphi \mapsto (-\log) \circ \varphi,$$

define $\exp: N_{\mathbb{R}} \rightarrow \mathbb{T}(\mathbb{R}_{\geq 0})$ by

$$\text{Hom}(M, \mathbb{R}) \rightarrow \text{Hom}(M, \mathbb{R}_{\geq 0}), \quad \varphi \mapsto \exp \circ (-\varphi).$$

These are inverses of each other. Although we overload the meanings of $\exp$ and $\log$, it should be clear from context which is intended. They fit into the

following commutative diagram, where val is defined to be the composition $\log \circ \rho_0$.

$$\begin{array}{ccc} \mathbb{T}^{\text{an}} & & \\ \downarrow \rho_0 & \searrow \text{val} & \\ \mathbb{T}(\mathbb{R}_{\geq 0}) & \xrightleftharpoons[\exp]{\log} & \mathbb{N}_{\mathbb{R}} \end{array}$$

To the fan Σ , we can also construct topological space $N_{\Sigma} \supset N_{\mathbb{R}}$ that can be viewed as a partial compactification of $N_{\mathbb{R}}$, such that N_{Σ} is compact if Σ is complete. The map $\exp: N_{\mathbb{R}} \rightarrow \mathbb{T}(\mathbb{R}_{\geq 0})$ defined above can be extended to a homeomorphism $\exp: N_{\Sigma} \xrightarrow{\sim} X_{\Sigma}(\mathbb{R}_{\geq 0})$. Similarly, $\log$ and val also extend to $\log: X_{\Sigma}(\mathbb{R}_{\geq 0}) \xrightarrow{\sim} N_{\Sigma}$ and $\text{val}: X_{\Sigma}^{\text{an}} \rightarrow N_{\Sigma}$.

These are the only facts we will use about N_{Σ} ; for details of the construction, see [BPS14, Section 4.1].

4.2 Analytic toric actions

Recall that we defined the compact torus $S \subset \mathbb{T}^{\text{an}}$ to be

$$S = \{p \in \mathbb{T}^{\text{an}} \mid |\chi^m(p)| = 1 \quad \forall m \in M\}.$$

In the archimedean case, recalling the $\mathbb{T}^{\text{an}}$ -action on $X_{\Sigma}^{\text{an}} = X_{\Sigma}(\mathbb{C})$ and the $\mathbb{T}(\mathbb{R}_{\geq 0})$ -action on $X_{\Sigma}(\mathbb{R}_{\geq 0})$ (the latter described in 4.1), we see that the map $\rho_{\Sigma}: X_{\Sigma}(\mathbb{C}) \rightarrow X_{\Sigma}(\mathbb{R}_{\geq 0})$ defined in the last section is torus-invariant: for all $t \in \mathbb{T}^{\text{an}} \simeq (\mathbb{C}^{\times})^n$ and $p \in X_{\Sigma}(\mathbb{C})$,

$$\rho_{\Sigma}(t \cdot p) = \rho_{\Sigma}(t) \cdot \rho_{\Sigma}(p).$$

Note that since $\rho_{\Sigma}(\mathbb{T}^{\text{an}}) \subset \mathbb{T}(\mathbb{R}_{\geq 0})$ by construction, the right-hand side is well-defined.

Moreover, by definition, $\rho_{\Sigma}(s) = 1$ for any $s \in S$, so $\rho_{\Sigma}(s \cdot p) = \rho_{\Sigma}(p)$. It moreover follows from the definition of $(\mathbb{C}^{\times})^n$ -action on $X_{\Sigma}(\mathbb{C})$ that

$$S \cdot p = \rho_{\Sigma}^{-1}(\rho_{\Sigma}(p)),$$

and the S -orbits of $X_{\Sigma}(\mathbb{C})$ are simply the fibres of ρ_{Σ} , as promised.

We will be interested in integrating over S -orbits, so let us remark that since $S \simeq (S^1)^n$ is a compact topological group, it admits a Haar measure of total volume 1.

The non-archimedean case is much more subtle. We will not go into a detailed discussion, but will give some highlights of what will be useful later on.

Let K be a non-archimedean field. In what follows, everything is assumed to be over K . Let G be an analytic group. The action of G on an analytic space X is defined as a morphism

$$\mu: G \times X \rightarrow X,$$

where $G \times X$ is the product in the category of analytic spaces over K .

Much like in an algebraic group scheme, the underlying topological points of G themselves do not form a group. Similarly, we cannot speak of $\mu(g, p)$, since $G \times X$ is not equal to the set-theoretic product $G \times_{\text{Set}} X$. However, there is a projection $\text{pr}: G \times X \rightarrow G \times_{\text{Set}} X$, and we can define

$$g \cdot p = \mu(\text{pr}^{-1}(g, p)).$$

However, now $g \cdot p$ ceases to be a point and is instead a *set* of points in X . But we can isolate a distinguished point $g * p$ in the set $g \cdot p$, if either $g \in G$ or $p \in X$ is *peaked*.

We will not define the property of being peaked.

Fact 4.1. Rational points of an analytic space are peaked. If either $g \in G$ or $p \in X$ is peaked, then $g * p$ is peaked.

We now return to the case of the $\mathbb{T}^{\text{an}}$ -action on X_{Σ}^{an} . We also have the torus-equivariance of ρ_{Σ} in the non-archimedean case.

Proposition 4.2. Let $t \in \mathbb{T}^{\text{an}}, p \in X_{\Sigma}^{\text{an}}$. Then

$$\rho_{\Sigma}(t \cdot p) = \rho_{\Sigma}(t) \cdot \rho_{\Sigma}(p).$$

In particular, ρ_{Σ} collapses the set $t \cdot p$ to a singleton.

The compact torus $S \subset \mathbb{T}^{\text{an}}$ is an analytic group whose underlying topological space is compact. However, it is not a topological group, so we cannot directly apply the theory of compact topological groups to obtain a Haar measure on S . Instead, we will identify a special point $\zeta \in S$, called the Gauss point, which can be thought of as the “generic point” of S ; and use the Dirac delta measure δ_{ζ} centred at ζ in place of the Haar measure.

Definition 4.3. The Gauss norm on $K[M]$ is defined by

$$\left\| \sum a_m \chi^m \right\| = \max |a_m|.$$

This is a multiplicative norm extending the absolute value on K , and so it corresponds to a point $\zeta \in \mathbb{T}^{\text{an}}$.

The following properties of the Gauss point illustrate why it can be thought of as the “generic point” of S .

Proposition 4.4. The Gauss point ζ is peaked, and $s * \zeta = \zeta$ for any $s \in S$. Moreover, $\zeta \cdot \zeta = S$.

In the non-archimedean case, we similarly expect the S -orbits are given by the fibres of ρ_{Σ} :

$$S \cdot p = \rho_{\Sigma}^{-1}(\rho_{\Sigma}(p)).$$

By the torus-equivariance of ρ_{Σ} (Proposition 4.2), we have $S \cdot p \subset \rho_{\Sigma}^{-1}(\rho_{\Sigma}(p))$. To show the other inclusion, we need to better understand the fibres of ρ_{Σ} .

Since we expect the fibres to be of the form $S \cdot p$, we may expect to find a point on each fibre with properties similar to those of the Gauss point ζ . The action by that point will allow us to connect different points on a fibre by an S -action.

Indeed, this is the case.

Proposition 4.5. There exists a continuous section of ρ_Σ , denoted

$$\theta_\Sigma: X_\Sigma(\mathbb{R}_{\geq 0}) \rightarrow X_\Sigma^{\text{an}},$$

such that $\rho_\Sigma(\gamma)$ is peaked for all $\gamma \in X_\Sigma(\mathbb{R}_{\geq 0})$. Denoting its restriction to the group $\mathbb{T}(\mathbb{R}_{\geq 0}) = U_0$ by θ_0 , we have $\theta_0(1) = \zeta$.

We moreover have the following properties analogous to Proposition 4.4.

Proposition 4.6. Suppose $t \in \mathbb{T}^{\text{an}}$, $p \in X_\Sigma^{\text{an}}$, and $\tau \in \mathbb{T}(\mathbb{R}_{\geq 0})$, $\gamma \in X_\Sigma(\mathbb{R}_{\geq 0})$. Then

$$\begin{aligned} t * \theta_\Sigma(\gamma) &= \theta_\Sigma(\rho_\Sigma(t) \cdot \gamma), \\ \theta_0(\tau) * p &= \theta_\Sigma(\tau \cdot \rho_\Sigma(p)). \end{aligned}$$

Now we can prove $S \cdot p \supset \rho_\Sigma^{-1}(\rho_\Sigma(p))$. Suppose $\rho(p) = \rho(q) = \gamma$. Then $\theta_\Sigma(1 \cdot \rho(p)) = \theta_\Sigma(1 \cdot \rho(q))$, so by the second equation above, we have $\theta_\Sigma(\gamma) = \zeta * p = \zeta * q$. So $S \cdot p \cap S \cdot q \neq \emptyset$, whence $q \in S \cdot p = S \cdot q$.

We have proven

Proposition 4.7. Let K be a non-archimedean field, and $p \in X_\Sigma^{\text{an}}$. Then

$$S \cdot p = \rho_\Sigma^{-1}(\rho_\Sigma(p)).$$

4.3 Toric metrics

We now want to define toric metrics as the class of continuous metrics which are invariant under the S -actions. First we recall some basic facts about divisors and line bundles on toric varieties.

Let Σ be a complete fan, and X_Σ be the associated complete toric variety. Recall that $\mathbb{T}$ -Cartier divisors (i.e. those invariant under $\mathbb{T}$ -actions) on X_Σ are classified by *virtual support functions* Ψ on Σ . A virtual support function $\Psi: |\Sigma| = \mathbb{N}_\mathbb{R} \rightarrow \mathbb{R}$ is a piecewise linear function on $|\Sigma|$ such that for each $\sigma \in \Sigma$, there exists $m(\sigma) \in M$ such that $\Psi|_\sigma = m(\sigma)|_\sigma$. The corresponding $\mathbb{T}$ -Cartier divisor $D_\Psi = \{(\chi^{-m(\sigma)}, U_\sigma)\}_{\sigma \in \Sigma}$ is given by the rational function $\chi^{-m(\sigma)}$ on the affine open set U_σ . Two Cartier divisors D_Ψ and $D_{\Psi'}$ are linearly equivalent if and only if there exists $m \in M$ such that $\Psi = \Psi' + m$.

The Cartier divisor D_Ψ gives rise to a line bundle $\mathcal{O}(D_\Psi) \subset \mathcal{K}_{X_\Sigma}$ generated by $\chi^{m(\sigma)}$ on U_σ . Here, $\mathcal{K}_{X_\Sigma}$ is the constant sheaf of rational functions on X_Σ . The element $1 \in \mathcal{K}_{X_\Sigma}$ corresponds to a distinguished rational section, denoted s_Ψ , of $\mathcal{O}(D_\Psi)$ such that $\text{div}(s_\Psi) = D_\Psi$.

The excision sequence of class groups implies that any line bundle L on X_Σ is isomorphic to some $\mathcal{O}(D_\Psi)$ ([Ful93, Proposition 3.4]). A rational section s of L is called a *toric section* if $\text{div}(s)$ is a $\mathbb{T}$ -Cartier divisor.

Definition 4.8. Let $\bar{L} = (L, \|\cdot\|)$ be a metrised line bundle on X_Σ . The metric $\|\cdot\|$ is a *toric metric* if for any toric section s , the function

$$\mathbb{T}^{\text{an}} \rightarrow \mathbb{R}_{\geq 0}, \quad p \mapsto \|s(p)\|$$

is constant along fibres of ρ_Σ . Hence its logarithm factors through a map $\psi_{\|\cdot\|}: N_\mathbb{R} \simeq \mathbb{T}(\mathbb{R}_{\geq 0}) \rightarrow \mathbb{R}$.

Definition 4.9. Let $\bar{L} = (L, \|\cdot\|)$ be a metrised line bundle on X_Σ equipped with a toric metric $\|\cdot\|$. Choose D_Ψ such that $L \simeq \mathcal{O}(D_\Psi)$. Then we can define a function $\psi_{\|\cdot\|}: N_\mathbb{R} \rightarrow \mathbb{R}$ by

$$\psi(u) = \log \|s_\Psi(p)\|.$$

Here, p is any point in the fibre $\rho_\Sigma^{-1}(\exp(u))$.

To avoid confusion, we have a minor but important remark.

Remark 4.10. As stated, the definition of $\psi_{\|\cdot\|}$ not only depends on $\|\cdot\|$ but also on the choice of Ψ such that $\mathcal{O}(D_\Psi) \simeq L$. If this ambiguity arises, we may write $\psi_{\bar{L}, \Psi}$ in place of $\psi_{\|\cdot\|}$. But we shall often simply present a line bundle L in the form of $\mathcal{O}(D_\Psi)$, in which case this Ψ (instead of another Ψ' for which $\mathcal{O}(D_\Psi) \simeq \mathcal{O}(D'_{\Psi'})$) is the obvious and tacit choice of Ψ for the above definition of $\psi_{\|\cdot\|}$. ∇

The main result of this section is

Theorem 4.11. Let $L = \mathcal{O}(D_\Psi)$ be a line bundle on X_Σ . The assignment $\|\cdot\| \mapsto \psi_{\|\cdot\|}$ gives a bijection between the toric metrics $\|\cdot\|$ on L and continuous functions $\psi: N_\mathbb{R} \rightarrow \mathbb{R}$ such that $\psi - \Psi$ extends to a continuous function on N_Σ .

Proof. First, we show that given a toric metric $\|\cdot\|$, the function $\psi = \psi_{\|\cdot\|}$ as defined above satisfies the required extension property. Recall that $L = \mathcal{O}(D_\Psi)$ is equipped with a distinguished toric section s_Ψ such that $\text{div}(s_\Psi) = D_\Psi$. Recall also that if Ψ is defined by the character $m(\sigma) \in M$ on the cone σ , then $D_\Psi|_{U_\sigma} = \text{div}_{K(X_\Sigma)}(\chi^{-m(\sigma)})$. Here, we use the subscript $K(X_\Sigma)$ to emphasise that this is the divisor of a *rational function*, rather than of a section of any line bundle.

Combining the two, we know that $\text{div}(\chi^{m(\sigma)} s_\Psi)|_{U_\sigma} = 0$, so that $\chi^{m(\sigma)} s_\Psi$ is a regular and non-vanishing toric section of $\mathcal{O}(D_\Psi)$ on U_σ . Therefore, for $p \in U_\sigma^{\text{an}}$, $\|\chi^{m(\sigma)} s_\Psi(p)\|$ is a well-defined number, whilst for $p \in \mathbb{T}^{\text{an}}$,

$$\begin{aligned} \left\| \chi^{m(\sigma)} s_\Psi(p) \right\| &= |\chi^{m(\sigma)}(p)| \|s_\Psi(p)\| \\ \log \left\| \chi^{m(\sigma)} s_\Psi(p) \right\| &= \log |\chi^{m(\sigma)}(p)| + \log \|s_\Psi(p)\| \\ &= -(m(\sigma), u) + \psi(u) \\ &= (\psi - \Psi)(u), \end{aligned}$$

where in the penultimate equality we recalled that the map $N_{\mathbb{R}} \rightarrow \mathbb{T}(\mathbb{R}_{\geq 0})$ is given by $u \mapsto \exp(-u)$, and so $|\chi^{m(\sigma)}(p)| = \exp(-(m(\sigma), u))$.

In particular, this shows that $(\psi - \Psi)(u)$ extends to a function on N_{σ} , completing the proof of the first direction.

The proof of the other implication is omitted. $\square$

From this correspondence, there is a natural metric one can put on $L = \mathcal{O}(D_{\Psi})$: namely, the metric $\|\cdot\|$ such that $\psi_{\|\cdot\|} = \Psi$.

Definition 4.12. The *canonical metric* on $L = \mathcal{O}(D_{\Psi})$, denoted $\|\cdot\|_{\text{can}}$, is the metric such that $\psi_{\|\cdot\|_{\text{can}}} = \Psi$. It satisfies

$$\|s_{\Psi}(p)\| = \exp(\Psi(u)).$$

Despite Remark 4.10, the canonical metric $\|\cdot\|_{\text{can}}$ genuinely does not depend on the choice of D_{Ψ} representing L : if $L = \mathcal{O}(D_{\Psi}) \simeq \mathcal{O}(D_{\Psi'})$, then requesting $\psi_{\bar{L}, \Psi} = \Psi$ and $\psi_{\bar{L}, \Psi'} = \Psi'$ defines the same metric on $\bar{L}$. This is because if $\Psi' = \Psi + m$ for some $m \in M$, then $s_{\Psi'} = \chi^m s_{\Psi}$, and so requiring $\|s_{\Psi}(p)\| = \exp(\Psi(u))$ and $\|s_{\Psi'}(p)\| = \exp(\Psi'(u))$ are the same.

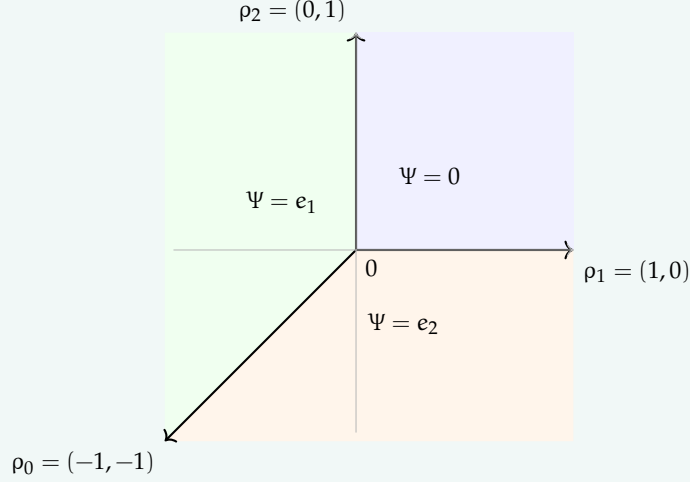
We will write $\bar{L}^{\text{can}} = (L, \|\cdot\|_{\text{can}})$ for L equipped with the canonical metric.

In the next section, we will learn that $\|\cdot\|_{\text{can}}$ is semipositive when Ψ is concave; equivalently, when $\mathcal{O}(D_{\Psi})$ is globally generated. It is in general not smooth (over archimedean fields K). But it is algebraic, when K is non-archimedean: its model is given by a *toric scheme over the DVR* K° . Toric schemes provide a natural class of models for toric metrics, called *toric models*, and there is a beautiful polyhedral-geometric classification of toric metrics and toric models in terms of recession functions and recession fans (see [BPS14, Section 4.5]).

While we do not have the space to develop the theory of toric models, we can give an explicit computation to show that the canonical metric on $(\mathbb{P}_K^n, \mathcal{O}(1))$, which was algebraic by definition, is a canonical metric in the present sense.

Example 4.13. Let $X = \mathbb{P}_K^n$, $L = \mathcal{O}(1)$. Recall from toric geometry that the polytope giving rise to X is the n -simplex $\Delta = \text{conv}(0, e_1, \dots, e_n)$, where $e_1, \dots, e_n$ is the standard basis of M . Let $\Psi = \Psi_{\Delta}$ be the support function of Δ . Then $\mathcal{O}(D_{\Psi}) = \mathcal{O}(1)$.

For notational simplicity, assume $n = 2$. Then Ψ is the piecewise affine function defined by $0, e_1, e_2$, respectively, in the three 2-dimensional cones as shown below.



The divisor D_Ψ can be computed by evaluating Ψ against the primitive generators of the three rays, and we find $D_\Psi = H_0$, where the hyperplanes H_0, H_1, H_2 correspond to the rays ρ_0, ρ_1, ρ_2 .

Choose coordinates $[x_0 : x_1 : x_2]$ on $\mathbb{P}^2$ such that $V(x_i) = H_i$. Then $s_\Psi = x_0$. Now, the canonical metric on X as defined in Example 3.27 gives

$$\|s_\Psi(x)\|_{\text{can}} = \frac{|x_0|}{\max_i |x_i|}.$$

Remembering $\psi_{\|\cdot\|_{\text{can}}}(u) = \log \|s_\Psi(x)\|_{\text{can}}$ and using $\log: X^{\text{an}} \rightarrow \mathbb{N}_{\mathbb{R}}$ on the right-hand side as discussed in Section 4.1 (noting the minus sign), we get a piecewise linear function $\psi_{\|\cdot\|_{\text{can}}}$ on $\mathbb{N}_{\mathbb{R}}$, which is none other than Ψ ! $\diamond$

4.4 Semipositive toric metrics

Our next goal is to examine the above correspondence in the case of semipositive metrics.

Let notations be as in the previous section. Let $L = \mathcal{O}(D_\Psi)$ be equipped with a toric metric $\|\cdot\|$ on X_Σ . Recall that for each toric metric $\|\cdot\|$ on L , we can associate a function $\psi_{\|\cdot\|}: \mathbb{N}_{\mathbb{R}} \rightarrow \mathbb{R}$.

Then the bijection given in the previous section can be refined as follows.

Theorem 4.14. The assignment $\|\cdot\| \mapsto \psi = \psi_{\|\cdot\|}$ is a bijection between *semipositive* toric metrics and *concave* functions on $\mathbb{N}_{\mathbb{R}}$ such that $\psi - \Psi$ is bounded on $\mathbb{N}_{\mathbb{R}}$.

Remark 4.15. The earlier condition on $\psi - \Psi$ was that it extends to a function to $\mathbb{N}_\Sigma$. This turns out to be equivalent to the boundedness condition here once ψ

is concave. ∇

To prove Theorem 4.14, we start by understanding better semipositive metrics which are themselves smooth or algebraic.

The archimedean case is a computation.

Proposition 4.16. Let $\|\cdot\|$ be a *smooth* toric metric on L . Then $\|\cdot\|$ is semipositive if and only if $\psi_{\|\cdot\|}$ is a *concave* function.

Proof Sketch. This comes down to a coordinate computation on the analytic torus $\mathbb{T}^{\text{an}} \subset X^{\text{an}}$.

Suppose $z_1, \dots, z_n$ are complex coordinates on $\mathbb{T}^{\text{an}}$. Then one can compute

$$\frac{1}{2\pi i} c_1(\bar{L}) = \frac{i}{\pi} \sum_{i,j} \frac{\partial^2 g}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j,$$

where $g = -\log \|s(z)\|$.

Then $\|\cdot\|^{\text{can}}$ is semipositive if and only if the matrix $G = \left(\frac{\partial^2 g}{\partial z_i \partial \bar{z}_j} \right)$ is positive semi-definite.

But letting $u_1, \dots, u_n$ be the coordinates on $N_{\mathbb{R}}$ corresponding to $z_1, \dots, z_n$ via the log map in Section 4.1, we have that ψ is concave if and only if

$$H = \text{Hess} \left(\frac{\partial^2 \psi}{\partial u_i \partial u_j} \right)$$

is negative semi-definite.

The two semi-definiteness conditions on the matrices are equivalent, once one unwinds the definitions. See [BPS14, Proposition 4.4.1] for details. $\square$

We have the analogous result in the non-archimedean case.

Proposition 4.17. Let $\|\cdot\|$ be a toric metric on L with a semipositive model. Then $\psi_{\|\cdot\|}$ is a *concave* function. Moreover, $\text{rec } \psi = \Psi$.

This is more difficult to prove, and we will omit the proof. The main difficulty is that a priori we have little control over which models the algebraic metric is induced from, so semipositivity is a somewhat elusive condition.

Using the above characterisations of smooth and algebraic semipositive metrics (Propositions 4.16 and 4.17), we can prove Theorem 4.14, by passing the above results to the limit.

Proof Sketch of Theorem 4.14. Suppose $\|\cdot\|$ is a semipositive toric metric, and $\psi = \psi_{\|\cdot\|}$. The boundedness of $\psi - \Psi$ follows immediately from the extension of $\psi - \Psi$ to a continuous function on the compact set N_{Σ} by Theorem 4.11.

We now show ψ is concave. By definition, there is a family of metrics $\|\cdot\|_t \rightarrow \|\cdot\|$, where each $\|\cdot\|_t$ is smooth and semipositive (in the archimedean case) or has a semipositive model (in the non-archimedean case). By an averaging process, we may take $\|\cdot\|_t$ to be toric.

Set $\psi_l = \psi_{\|\cdot\|_l}$. Then by Propositions 4.16 and 4.17, ψ_l is concave on $N_{\mathbb{R}}$. Then ψ is also concave because $\|\cdot\|_l \rightarrow \|\cdot\|$ implies that $\psi_l \rightarrow \psi$ uniformly.

The converse requires more than we had time to build and we can only give a brief sketch. Given a concave function ψ on $N_{\mathbb{R}}$ such that $\psi - \Psi$ is bounded, the proof constructs a sequence of *piecewise affine* concave functions ψ_l such that $\psi_l \rightarrow \psi$ uniformly. Next, we use that piecewise affine concave functions always correspond to semipositive metrics ([BPS14, Proposition 4.3.22]). Interestingly, in both the archimedean and non-archimedean cases, this is proven independently of the above Propositions 4.16 and 4.17; instead, it was by exhibiting such metrics as inverse images of the canonical metrics on projective spaces, which are semipositive.

Now take $\|\cdot\|, \|\cdot\|_l$ to be the metrics corresponding to ψ, ψ_l . Then $\|\cdot\| = \lim_{l \rightarrow \infty} \|\cdot\|_l$ is the limit of semipositive metrics, and so is semipositive. $\square$

4.5 Associated measures of toric metrics

Let Σ be a complete fan and let (X_{Σ}, D_{Ψ}) be as before. Let $\bar{L} = (\mathcal{O}(D_{\Psi}), \|\cdot\|)$ be a metrised toric line bundle equipped with a toric metric.

Recall from Chapter 3 that to define the height of a d -dimensional cycle Y on X_{Σ} with respect to metrised line bundles $\bar{L}_0, \dots, \bar{L}_{d-1}$, we need to integrate against the measure $c_1(\bar{L}_0) \wedge \dots \wedge c_1(\bar{L}_{d-1}) \wedge \delta_Y$.

It suffices, for the main results of this exposition, to consider the case $Y = X_{\Sigma}$, and $\bar{L}_0, \dots, \bar{L}_{n-1} = \bar{L}$. So

$$c_1(\bar{L}_0) \wedge \dots \wedge c_1(\bar{L}_{d-1}) \wedge \delta_Y = c_1(\bar{L})^{\wedge n} \wedge \delta_{X_{\Sigma}}.$$

Since $X_{\Sigma}^{\text{an}} \setminus \mathbb{T}^{\text{an}}$ has measure zero with respect to $c_1(\bar{L})^{\wedge n} \wedge \delta_{X_{\Sigma}}$, for the purpose of height computations, it makes sense to restrict our attention to $\mathbb{T}^{\text{an}}$ and ask how we can transport the height computation to $N_{\mathbb{R}}$ via $\text{val}: \mathbb{T}^{\text{an}} \rightarrow N_{\mathbb{R}}$.

For a simpler statement, recall that val can be extended to $X_{\Sigma}^{\text{an}} \rightarrow N_{\Sigma}$. Then, surprisingly, it turns out that $\text{val}_*(c_1(\bar{L})^{\wedge n} \wedge \delta_{X_{\Sigma}})$ is a Monge-Ampère measure on N_{Σ} , in the following sense.

Recall from Section 2.5 the definition of the Monge-Ampère measure $\mathcal{M}(\psi)$ associated to a concave function $\psi: N_{\mathbb{R}} \rightarrow \mathbb{R}$. By abuse of notation, denote also by $\mathcal{M}(\psi)$ its extension to N_{Σ} by zero: for $E \subset N_{\mathbb{R}}$, set

$$\mathcal{M}(\psi)(E) = \mathcal{M}(\psi)(E \cap N_{\mathbb{R}}).$$

Then we have

Theorem 4.18. Suppose X_{Σ} is defined over an archimedean field or a discretely valued non-archimedean field K , and $\bar{L} = (\mathcal{O}(D_{\Psi}), \|\cdot\|)$ is equipped with a *semi-positive* toric metric $\|\cdot\|$. Then

$$\text{val}_*(c_1(\bar{L})^{\wedge n} \wedge \delta_{X_{\Sigma}}) = n! \mathcal{M}(\psi_{\|\cdot\|}).$$

The proof is by first proving the result for smooth semipositive metrics (if K is archimedean) or metrics with a semipositive model (if K is non-archimedean), and then passing to the limit according to the definition of semipositive metrics.

Neither of these latter results is very hard to prove: the first is by an explicit computation in coordinates, similar to the proof of Proposition 4.16; the second uses the classification of toric models in terms of rational piecewise affine functions and then the fact that the Monge-Ampère measures of such functions are supported on the vertices of a polyhedral complex. For details, see [BPS14, Theorems 4.4.4, 4.7.4].

Chapter 5

Toric Heights

5.1 Local toric heights

Let Σ be a complete fan on $N_{\mathbb{R}}$, and let X_{Σ} be the corresponding complete toric variety. Let $\Psi: N_{\mathbb{R}} \rightarrow \mathbb{R}$ be a *support function* on Σ , i.e. a virtual support function (see the beginning of Section 4.3) on Σ which is concave. In this case, Ψ is the support function of the polytope

$$\Delta_{\Psi} := \text{stab}(\Psi),$$

which is the lattice polytope $\text{conv}(m(\sigma))_{\sigma \in \Sigma}$. The corresponding line bundle $L = \mathcal{O}(\mathbb{D}_{\Psi})$ is globally generated.

Let $\bar{L} = (\mathcal{O}(\mathbb{D}_{\Psi}), \|\cdot\|)$ be equipped with a semipositive toric metric.

Recall that by Theorem 4.14, semipositivity implies that the associated function $\psi = \psi_{\|\cdot\|}: N_{\mathbb{R}} \rightarrow \mathbb{R}$ is concave.

Our first observation is that $\text{stab}(\psi_{\|\cdot\|})$ does not depend on the metric chosen.

Lemma 5.1. We have

$$\text{stab}(\psi) = \text{stab}(\Psi).$$

Proof. Recall from Theorem 4.14 that $\psi - \Psi$ is bounded on $N_{\mathbb{R}}$. The result then follows from

$$\inf_{u \in N_{\mathbb{R}}} (m, u) - \psi(u) = -\infty \iff \inf_{u \in N_{\mathbb{R}}} (m, u) - \Psi(u) = -\infty.$$

□

Let $\Delta_{\Psi} = \text{stab}(\Psi)$ be the polytope determined by Ψ .

Definition 5.2. Let $\bar{L}$ be as above. The *roof function* associated to $\bar{L}$ is defined to be the Legendre-Fenchel dual of $\psi_{\|\cdot\|}$, denoted by

$$\vartheta_{\|\cdot\|} = \psi_{\|\cdot\|}^{\vee}: \Delta_{\Psi} \rightarrow \mathbb{R}.$$

This is well-defined by the above lemma.

Similar to when we defined global heights, we now make the simplifying assumption that X_Σ is *projective*.

Definition 5.3. Let $\{\bar{L}_i = (L_i, \|\cdot\|_i)\}_{0 \leq i \leq d}$ be a family of toric line bundles with semipositive toric metrics. Recall that each L_i can also be equipped with the canonical metric; denote with $\bar{L}_i^{\text{can}}$ the corresponding metrised line bundle.

Let $Y \subset X_\Sigma$ be a d -dimensional irreducible subvariety. We define the *toric local height* of Y with respect to $\{\bar{L}_i\}$ to be the following difference of local heights

$$h_{\bar{L}_0, \dots, \bar{L}_d}^{\text{tor}}(Y) = h_{\bar{L}_0, \dots, \bar{L}_d}(Y; s_0, \dots, s_d) - h_{\bar{L}_0^{\text{can}}, \dots, \bar{L}_d^{\text{can}}}(Y; s_0, \dots, s_d),$$

where $s_0, \dots, s_d$ are sections of $L_0, \dots, L_d$ meeting Y properly. The advantage of this definition is that it does not depend on the choice of sections.

We extend the definition to d -dimensional cycles by linearity.

Remark 5.4. The toric local height is not a local height in the sense of Section 3.3. The most important reason for which we make the current definition is that it is independent of the choice of sections. For one thing, this independence is crucial in the inductive proof of the local toric height formula (Theorem 5.5) below: when we restrict $\bar{L}$ to any of the prime divisors $V(\rho_F)$ in the support of $\div(s_\Psi)$, the section $s_\Psi|_{V(\rho_F)}$ can no longer be used to compute the concave function associated to $\bar{L}|_{V(\rho_F)}$, and we are forced to work with a different section.

However, in the next section, we will discuss how this can still allow us to compute global heights in the sense of Section 3.4. ∇

When X_Σ is not projective, the local toric height of Y can still be defined by choosing a birational projective $\varphi: Y' \rightarrow Y$, and replace Y by Y' , $\bar{L}_i$ by $\varphi^* \bar{L}_i$, $\bar{L}_i^{\text{can}}$ by $\varphi^* \bar{L}_i^{\text{can}}$ in the right-hand side of the above equality.

In fact, by an inclusion-exclusion-type formula, we can restrict our attention to the case where all $\bar{L}_i = \bar{L}$. In this case, denote

$$h_{\bar{L}}^{\text{tor}}(Y) = h_{\bar{L}_0, \dots, \bar{L}_d}^{\text{tor}}(Y).$$

We now come to the main theorem of the entire exposition.

Theorem 5.5. Let Σ be a complete fan on $N_{\mathbb{R}}$, and let $X = X_\Sigma$ be the corresponding complete toric variety. Let $\Psi: N_{\mathbb{R}} \rightarrow \mathbb{R}$ be a *support function* on Σ , and let D_Ψ be the corresponding $\mathbb{T}$ -Cartier divisor. Let $\bar{L} = (\mathcal{O}(D_\Psi), \|\cdot\|)$ be equipped with a semipositive metric $\|\cdot\|$, with corresponding concave function $\psi = \Psi_{\|\cdot\|}$. Put $\Delta = \text{stab}(\Psi)$ and $\vartheta = \psi^\vee$ as before. Then

$$h_{\bar{L}}^{\text{tor}}(X) = (n+1)! \int_{\Delta} \vartheta \, d\text{vol}_M.$$

We will prove this in the case when Ψ is strictly concave, or equivalently, L is ample. The general case begins by reducing to this case, noting that L is already globally generated and then using the general theory of toric varieties.

We separate a few notations and lemmas to streamline the proof.

Suppose $F \subset \Delta$ is a facet of Δ . Let $\rho_F \in \Sigma^1$ be the corresponding ray in the fan Σ , and let v_F be the primitive vector of ρ_F , i.e. the minimal integral vector along ρ_F .

Define $M(F)$ to be the rank- $(n-1)$ sublattice $M \cap v_F^\perp$ of M , and let $N(F)$ be the corresponding dual lattice. Note that we may regard F as a lattice polytope in $M(F)_\mathbb{R}$, up to the ambiguity of translation. Indeed, for any $m_0 \in F \cap M$ or more generally $m_0 \in \text{aff}(F) \cap M$, the translation

$$\text{aff}(F) \xrightarrow{\sim} M_\mathbb{R}, \quad m \mapsto m - m_0$$

defines an inclusion $F \hookrightarrow M(F)_\mathbb{R}$ as a lattice polytope, with $M(F)_\mathbb{R}$ considered as a subspace of $M_\mathbb{R}$.

Now let $i_F: X_F = V(\rho_F) \rightarrow X$ be the inclusion of the divisor corresponding to F . Let Ψ_F be a support function on $N(F)_\mathbb{R}$ such that $i_F^* L \simeq \mathcal{O}_{X_F}(D_{\Psi_F})$. Then we can speak of the associated concave function $\psi_{i_F^* L, \Psi_F}$ of the pullback metric on $i_F^* \bar{L}$ and the corresponding roof function $\vartheta_{i_F^* \bar{L}, \Psi_F}$. Note that Ψ_F is not uniquely determined (cf. Remark 4.10).

Then we have the following functorial property of metrised toric line bundles and associated roof functions.

Lemma 5.6. Let $\vartheta_{i_F^* \bar{L}, \Psi_F}$ be as above. Then the choice of Ψ_F is in fact equivalent to a choice of $m_0 \in F \cap M$, determining an inclusion $F \hookrightarrow M(F)_\mathbb{R}$ as above. Under this identification, we have $\text{dom}(\vartheta_{i_F^* \bar{L}, \Psi_F}) = F$, and moreover,

$$\vartheta_{i_F^* \bar{L}, \Psi_F} = \vartheta|_F.$$

This is a simplified rephrase of [BPS14, Proposition 4.8.9].

We also need the following variant of the Stokes-type formula for Monge-Ampère measures (Theorem 2.26).

Proposition 5.7. Let $f: N_\mathbb{R} \rightarrow \mathbb{R}$ be a concave function such that $\text{stab}(f)$ is equal to the polytope $\Delta = \text{stab}(\Psi)$ above.

Then

$$-\int_{N_\mathbb{R}} f \, d\mathcal{M}_M(f) = (n+1) \int_\Delta f^\vee \, d\text{vol}_M + \sum_F \Psi(v_F) \int_F f^\vee \, d\text{vol}_{M(F)}.$$

Proof. We choose a basis $(m_1, \dots, m_n)$ of M such that $(m_2, \dots, m_n)$ is a basis of $M(F)$ and $m_1 = -v_F$ points out of Δ . Expressing λ in this basis we obtain

$$\lambda|_F = -\Psi(v_F) \, d\text{vol}_{M(F)}.$$

The result then follows from Theorem 2.26. $\square$

Proof. Now we proceed by induction on $n = \dim X$. Denote $s = s_\Psi$.

If $n = 0$, then $\Psi = 0, \Delta = \{0\}$, and X is a point p . Recalling the definitions of ψ and the canonical metric (Definitions 4.9 and 4.12), we have $\log \|s(p)\| = \psi(0), \log \|s(p)\|_{\text{can}} = \Psi(0) = 0$. Recall also that the height of a point is simple:

$$\begin{aligned} h^{\text{tor}}(X) &= h_{\bar{L}}(p) - h_{\bar{L}^{\text{can}}}(p) \\ &= -\log \|s(p)\| + \log \|s(p)\|_{\text{can}} \\ &= -\psi(0). \end{aligned}$$

But $\vartheta(0) = -\psi(0)$, and so

$$h^{\text{tor}}(X) = -\psi(0) = \int_{\Delta} \vartheta \, d\text{vol}_{\mathcal{M}}$$

when $\dim X = 0$.

Now assume $n > 0$. Choose rational sections $s_0, \dots, s_{n-1}$ of L such that $s_0, \dots, s_{n-1}, s$ intersect X properly. By the inductive definition of local heights, we have

$$h_{\bar{L}}(X; s_0, \dots, s_{n-1}, s) = h_{\bar{L}}(\text{div}(s); s_0, \dots, s_{n-1}) - \int_{X^{\text{an}}} \log \|s\| \, c_1(\bar{L})^{\wedge n} \wedge \delta_{X_{\Sigma}}.$$

Now, recall from toric geometry that $\text{div}(s) = \sum_F -\Psi(v_F) V(\rho_F)$. Therefore, the first term in the right-hand side above can be written as

$$h_{\bar{L}}(\text{div}(s); s_0, \dots, s_{n-1}) = - \sum_F \Psi(v_F) h_{\bar{L}}(V(\rho_F); s_0, \dots, s_{n-1}).$$

For the second term, recall from Theorem 4.18 that

$$\text{val}_*(c_1(\bar{L})^{\wedge n} \wedge \delta_{X_{\Sigma}}) = n! \mathcal{M}_{\mathcal{M}}(\psi),$$

and that by definition $\text{val}^* \psi = \log \|s\|$, so we have

$$\begin{aligned} \int_{X^{\text{an}}} \log \|s\| \, c_1(\bar{L})^{\wedge n} \wedge \delta_{X_{\Sigma}} &= \int_{X^{\text{an}}} \text{val}^* \psi \, c_1(\bar{L})^{\wedge n} \wedge \delta_{X_{\Sigma}} \\ &= n! \int_{N_{\mathbb{R}}} \psi \, d\mathcal{M}_{\mathcal{M}}(\psi). \end{aligned}$$

Repeating the same calculations for $h_{\bar{L}^{\text{can}}}(X; s_0, \dots, s_{n-1}, s)$, we have

$$\begin{aligned} h_{\bar{L}}^{\text{tor}}(X) &= \sum_F -\Psi(v_F) h_{\bar{L}}^{\text{tor}}(V(\rho_F)) - n! \int_{N_{\mathbb{R}}} \psi \, d\mathcal{M}_{\mathcal{M}}(\psi) \\ &\quad + n! \int_{N_{\mathbb{R}}} \Psi \, d\mathcal{M}_{\mathcal{M}}(\Psi) \end{aligned}$$

But now, since Ψ is piecewise affine, recall from Proposition 2.25 that $\mathcal{M}_{\mathcal{M}}(\Psi)$ is a discrete measure supported on the vertices of $\Pi(\Psi)$, which is simply $\{0\}$. But $\Psi(0) = 0$, so the last term in the above sum vanishes.

But now,

$$h_{\bar{L}}^{\text{tor}}(V(\rho_F)) = h_{i_F^* \bar{L}}^{\text{tor}}(V(\rho_F))$$

by general facts of functoriality of local heights, which is in turn equal to

$$\int_{\text{dom}(\vartheta_{i_F^* \bar{L}, \Psi_F})} \vartheta_{i_F^* \bar{L}, \Psi_F} \, d \text{vol}_{M(F)} = \int_F \vartheta|_F \, d \text{vol}_{M(F)}$$

by the inductive hypothesis and Lemma 5.6.

Hence

$$\begin{aligned} h_{\bar{L}}^{\text{tor}}(X) &= \sum_F -\Psi(v_F) \int_F (\vartheta|_F) \, d \text{vol}_{M(F)} - n! \int_{N_{\mathbb{R}}} \psi \, d\mathcal{M}_M(\psi) \\ &= (n+1)! \int_{\Delta} \vartheta \, d \text{vol}_M. \end{aligned}$$

□

Remark 5.8. One may reasonably ask whether this theorem allows us to compute the heights of other cycles. The general fact about the functoriality of local heights, appealed to in the above proof, tells us that if $Y \subset X$ is a toric subvariety, then

$$h_{\bar{L}}^{\text{tor}}(Y) = h_{\bar{L}|_Y}^{\text{tor}}(Y),$$

and this extends by linearity to cycles supported on toric subvarieties.

But in general, neither the theorem nor the proof tells us how to compute the heights of non-toric cycles, which should be expected. ▽

5.2 Global heights from local toric heights

We now return to the global setting. Let $(\mathbb{K}, \mathfrak{M})$ be a global field, and let $X = X_{\Sigma}$ be a complete toric variety over $\mathbb{K}$. Let Ψ be a support function on Σ , and let $L = \mathcal{O}(D_{\Psi})$ as before.

For each place $v \in \mathfrak{M}$, write $X_v = X \times_{\mathbb{K}} \text{Spec } \mathbb{K}_v$ and L_v for the pullback of L to X_v . We will work with a metrised line bundle

$$\bar{L} = (L, (\|\cdot\|_v)_{v \in \mathfrak{M}})$$

such that each $\|\cdot\|_v$ is a semipositive toric metric on L_v , and such that $\|\cdot\|_v = \|\cdot\|_{\text{can}, v}$ for all but finitely many places v . In particular, $\bar{L}$ is quasi-algebraic in the sense of Section 3.4, so the global height $h_{\bar{L}}(X)$ is well-defined.

Recall that the global height is obtained by summing the corresponding local heights:

$$h_{\bar{L}}(X) = \sum_{v \in \mathfrak{M}} n_v \, h_{\bar{L}_v}(X_v; s_0, \dots, s_n),$$

where $s_0, \dots, s_n$ are rational sections of L meeting X properly. Since $\mathbb{K}$ satisfies the product formula, the resulting global height is independent of the choice of these sections.

We will also write $\bar{L}^{\text{can}}$ for L equipped with the canonical metric at every place.

Theorem 5.9. Let $X = X_\Sigma$ and $L = \mathcal{O}(D_\Psi)$ be as above. Then

$$h_{\bar{L}^{\text{can}}}(X) = 0.$$

Proof Sketch. The proof is inductive and very similar to the proof of Theorem 5.5. The key observation is the vanishing of

$$n! \int_{N_{\mathbb{R}}} \Psi \, d\mathcal{M}_{\mathcal{M}}(\Psi)$$

in the recursive computation of the local height $h_{\bar{L}_v^{\text{can}}}(X_v; s_0, \dots, s_n)$, as in that proof. $\square$

The immediate corollary is that global heights can be computed by the weighted sum of *toric* local heights.

Corollary 5.10. Let $\bar{L}$ be as above. Then

$$h_{\bar{L}}(X) = \sum_{v \in \mathfrak{M}} n_v h_{\bar{L}_v}^{\text{tor}}(X_v).$$

Example 5.11. We now return to the Guiding Example 3.27:

Let $X = \mathbb{P}_{\mathbb{Q}}^n$ and $L = \mathcal{O}(1)$. Recall that the model $(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}(1))$ specifies the canonical metric on L at the non-archimedean places, which was first defined in Example 3.12. As we found in Example 4.13, this canonical metric agrees with the canonical toric metric.

Together with the Fubini-Study metric on L at the archimedean place, we have made L into a metrised line bundle on X , denoted by $\bar{L}_{\text{FS}}$. We now compute its global height.

At all finite places, the toric local height is zero. Hence

$$h_{\bar{L}_{\text{FS}}}(X) = h_{\bar{L}_{\text{FS}}}^{\text{tor}}(X_{\infty}).$$

It remains only to compute the concave function associated to the archimedean metric. We proceed similarly as in Example 4.13.

Let $\Delta = \text{conv}(0, e_1, \dots, e_n)$, and let Ψ be its support function. Then $L = \mathcal{O}(D_\Psi)$. Taking coordinates $[x_0 : \dots : x_n]$ so that $s_\Psi = x_0$, write a point of the dense torus as

$$[1 : z_1 : \dots : z_n],$$

and let $u_i = -\log|z_i|$ be the corresponding coordinates on $N_{\mathbb{R}}$. The

Fubini-Study metric gives

$$\|s_\Psi\|_{\text{FS}} = \frac{1}{(1 + |z_1|^2 + \dots + |z_n|^2)^{1/2}},$$

so, remembering that $\psi_{\text{FS}}(u) = \log \|s_\Psi\|_{\text{FS}}$,

$$\psi_{\text{FS}}(u) = -\frac{1}{2} \log(1 + e^{-2u_1} + \dots + e^{-2u_n}).$$

Thus, if $\vartheta_{\text{FS}} = \psi_{\text{FS}}^\vee$ is the associated roof function on Δ , the global height is

$$h_{\overline{L}_{\text{FS}}}(\mathbb{P}_{\mathbb{Q}}^n) = (n+1)! \int_{\Delta} \vartheta_{\text{FS}} \, d\text{vol}_{\mathcal{M}}.$$

◇

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